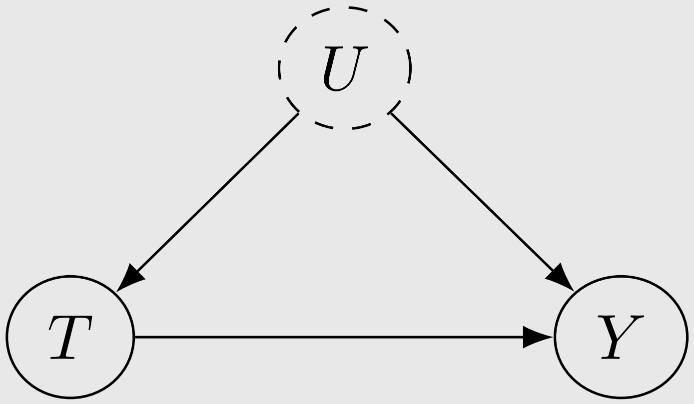
Instrumental Variables

Brady Neal

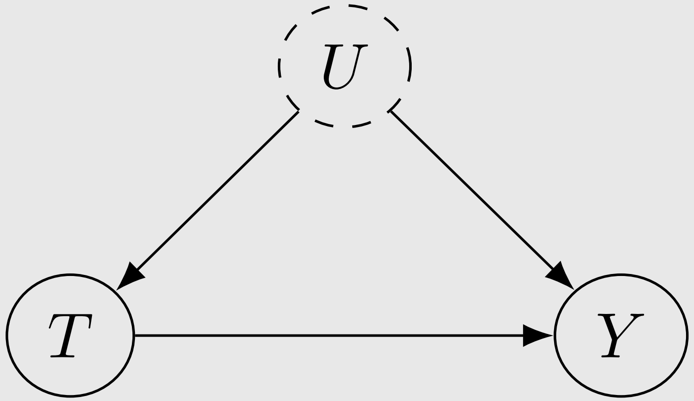
causalcourse.com

Unobserved Confounding



Unobserved Confounding

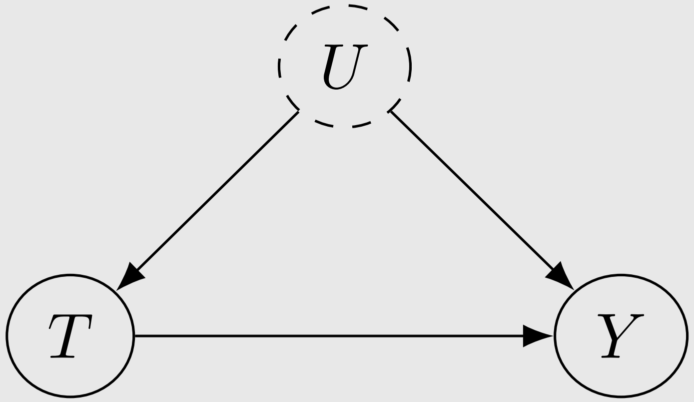
Week 5:



Unobserved Confounding

Week 5:

- Frontdoor adjustment



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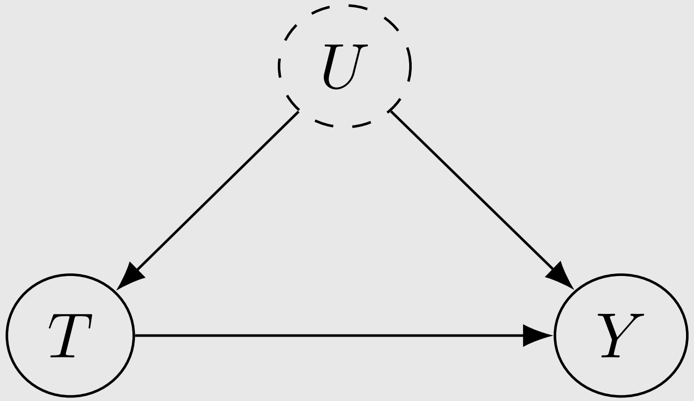

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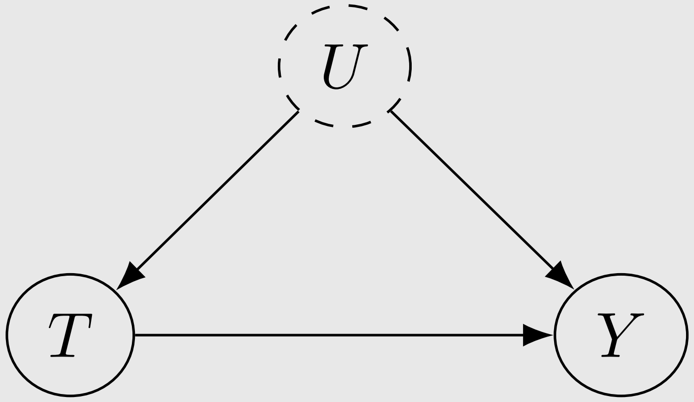
- Some other fancy application of do-calculus



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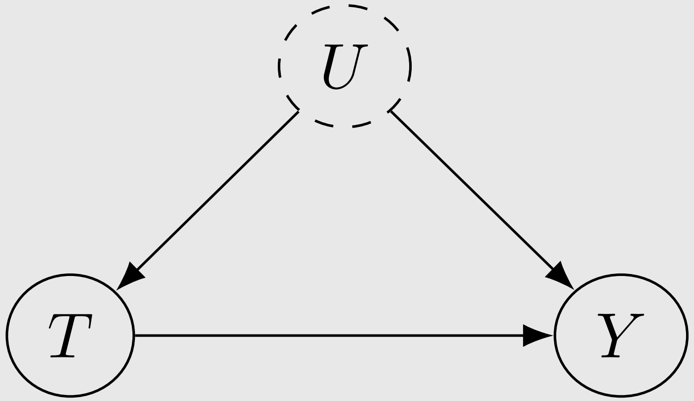

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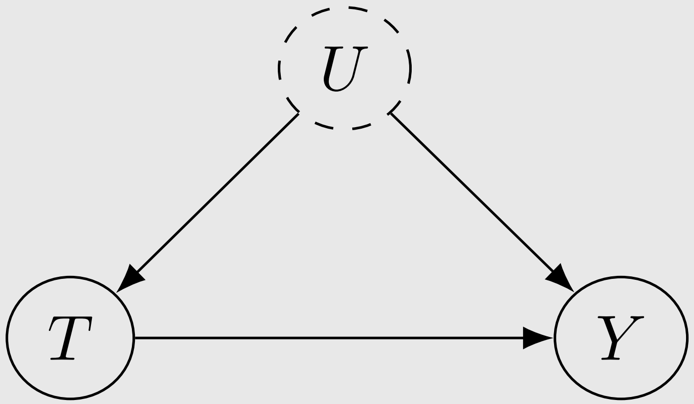

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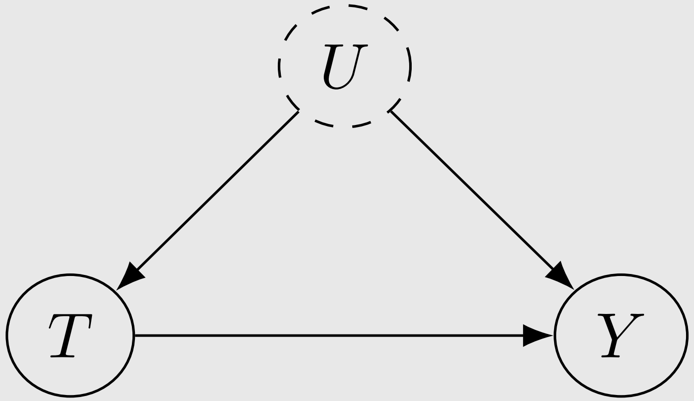

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This week:
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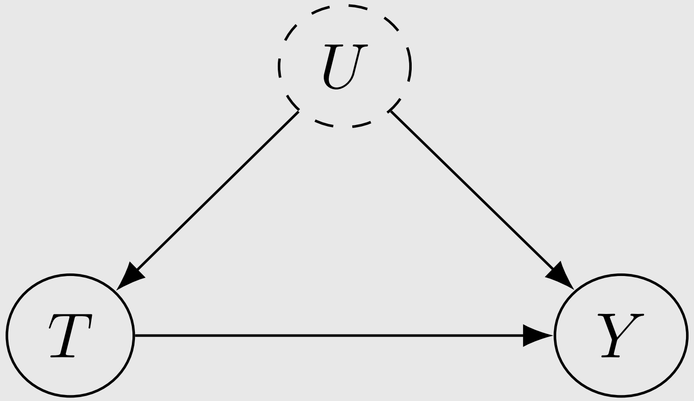

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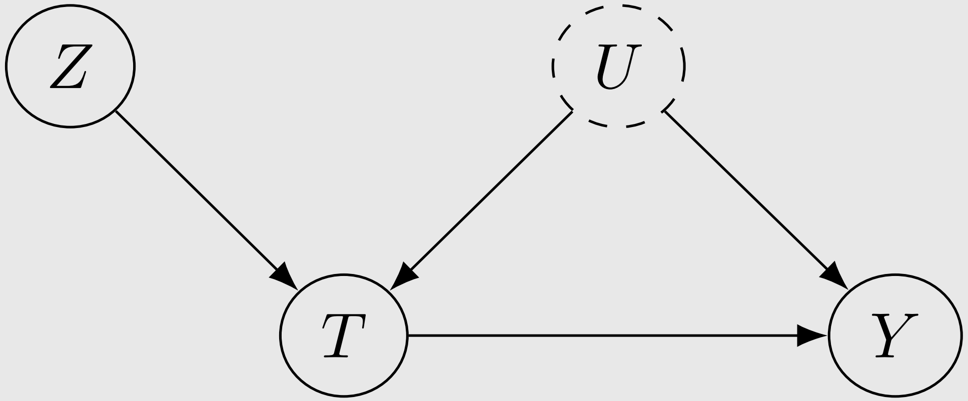

What is an Instrument?

No Nonparametric Identification of the ATE

Warm-Up: Linear Setting

Nonparametric Identification of Local ATE

More General Settings for the ATE

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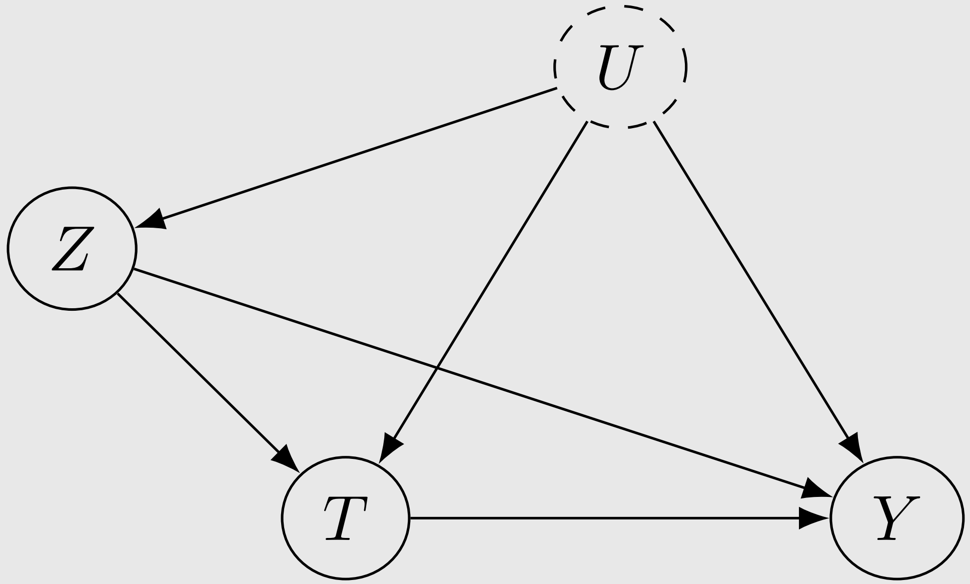
Assumption 1: Relevance

Z has a causal effect on T



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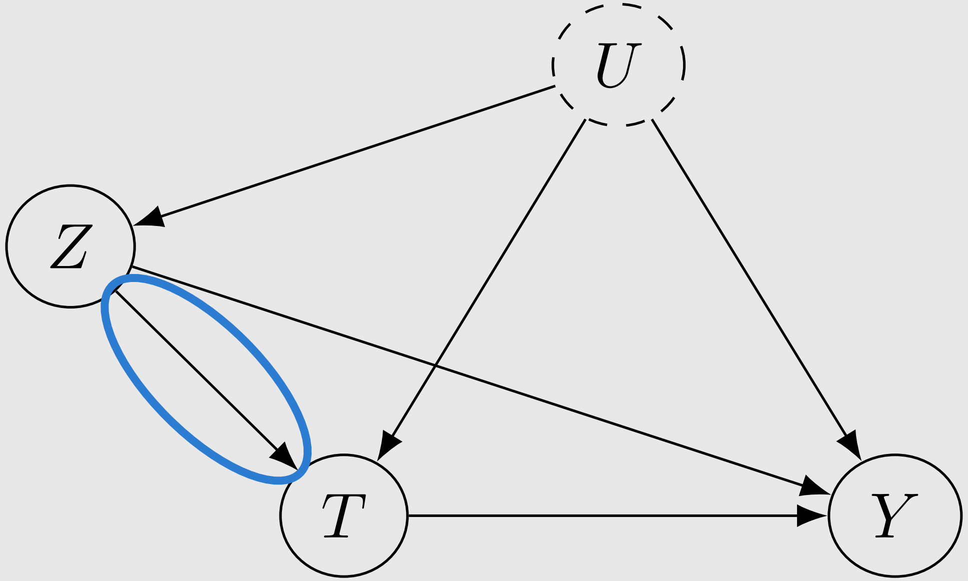
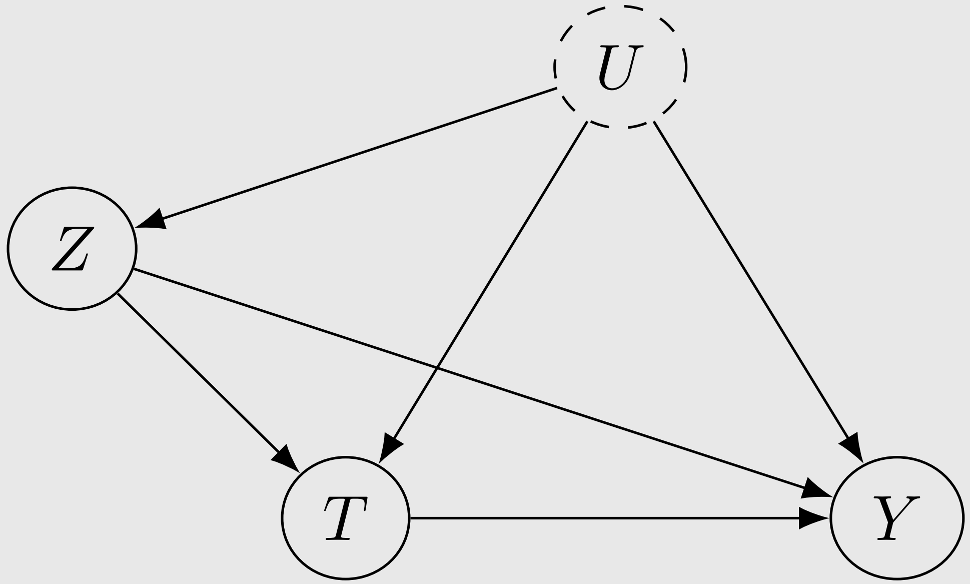


Assumption 2: Exclusion Restriction

The causal effect of Z on Y is fully mediated by T



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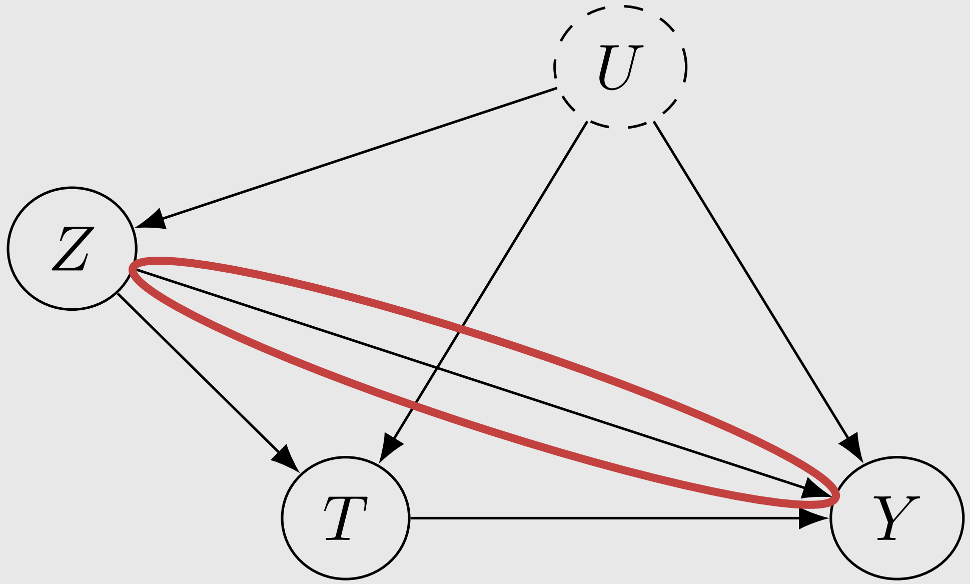

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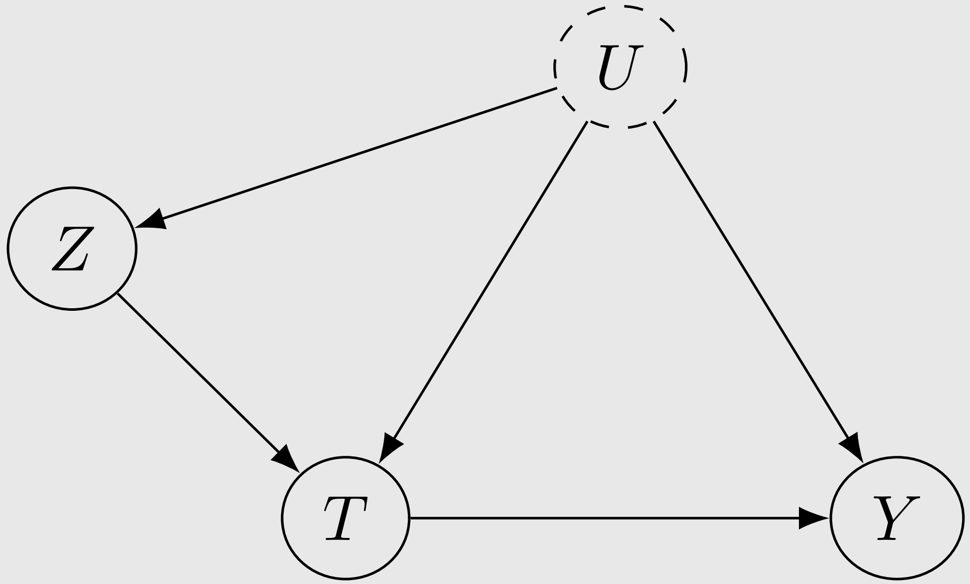

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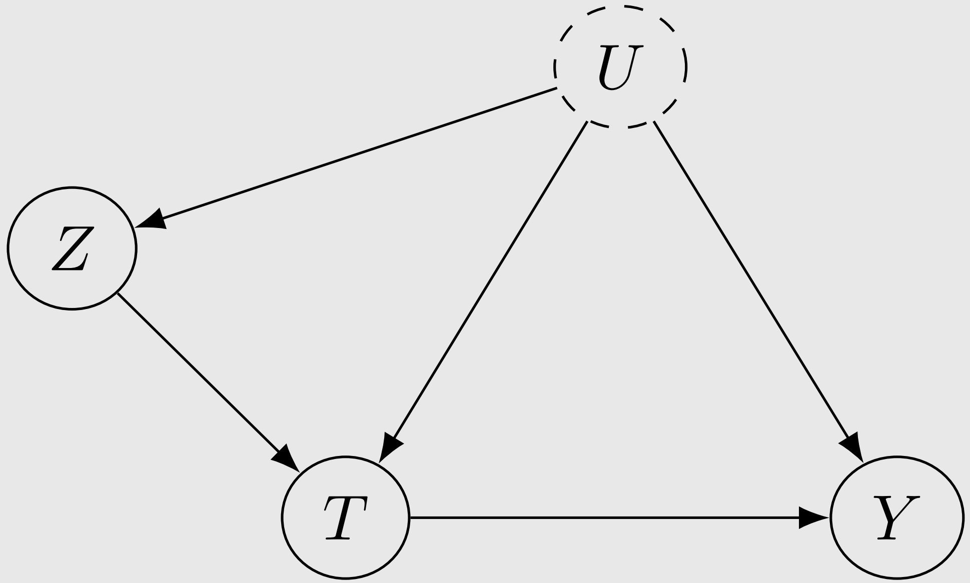
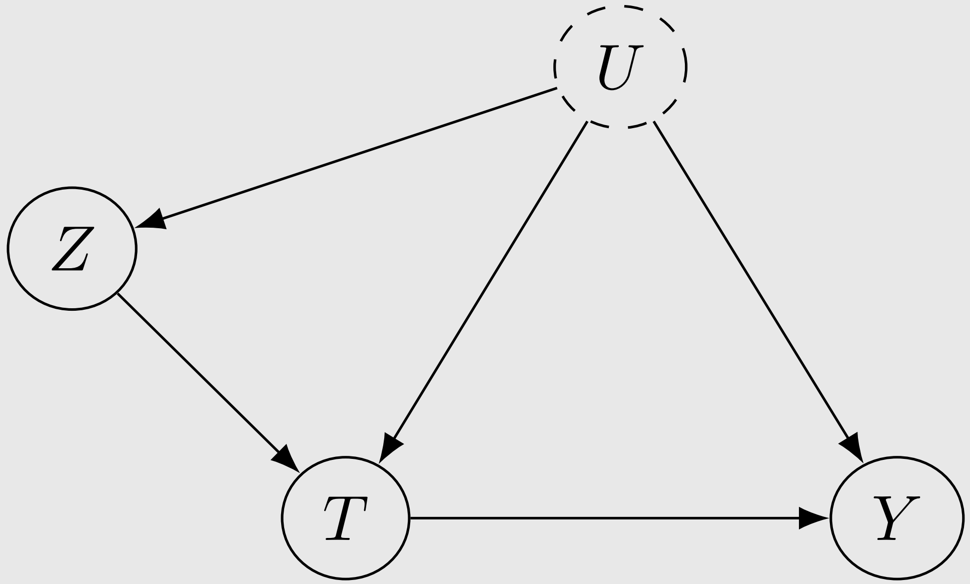


Recall:
Removing edges corresponds to adding assumptions

Assumption 3: Instrumental Unconfoundedness

Z is unconfounded (no unblockable backdoor paths to Y)



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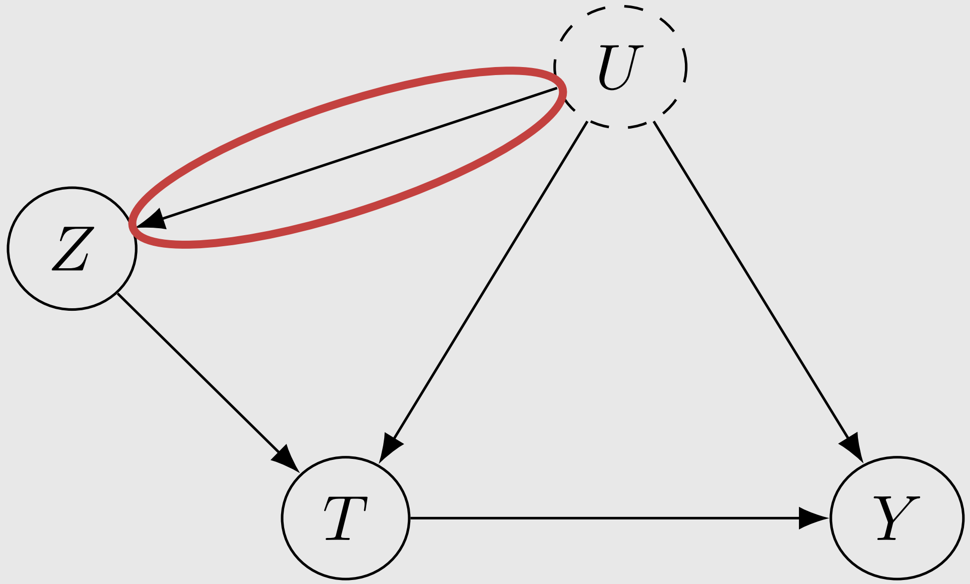

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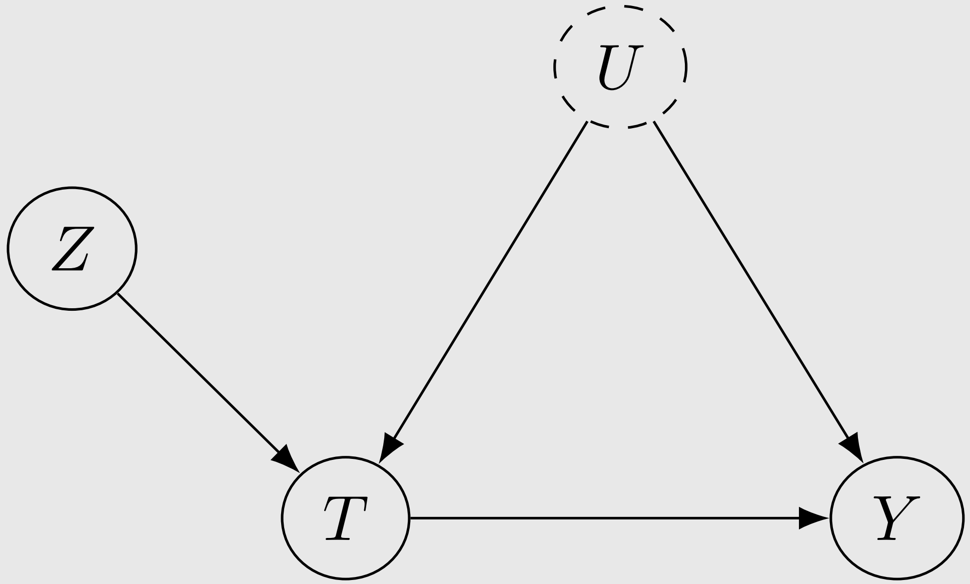
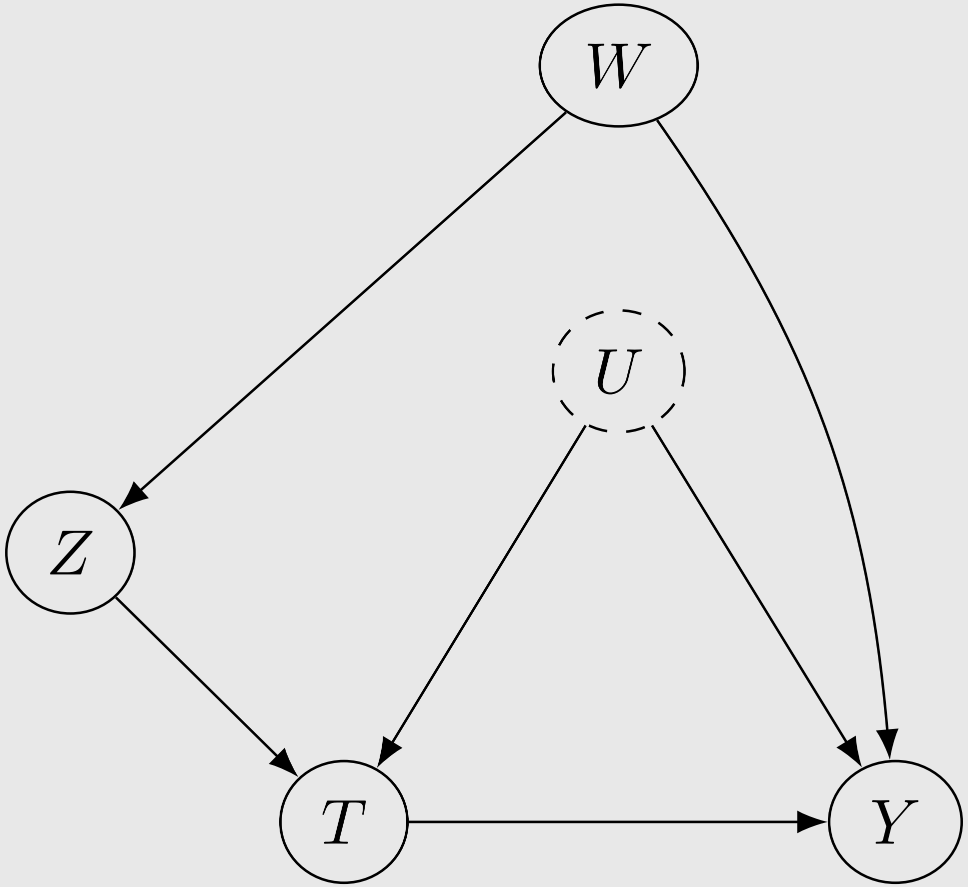


Conditional Instruments



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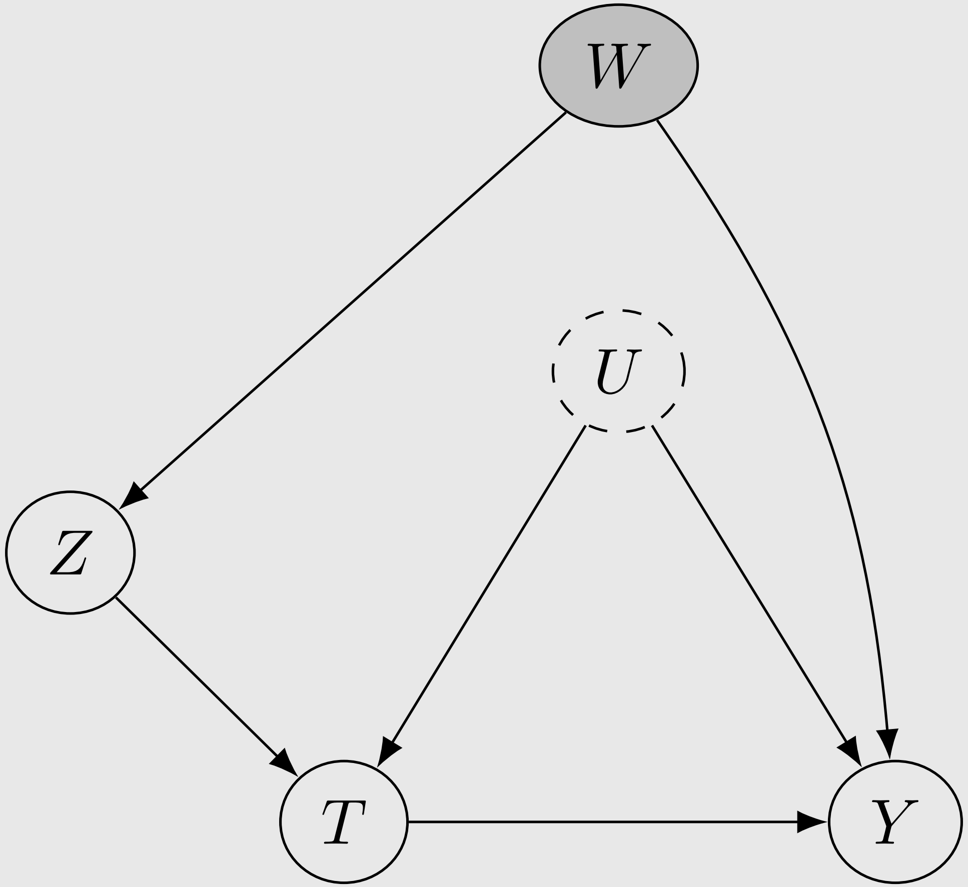
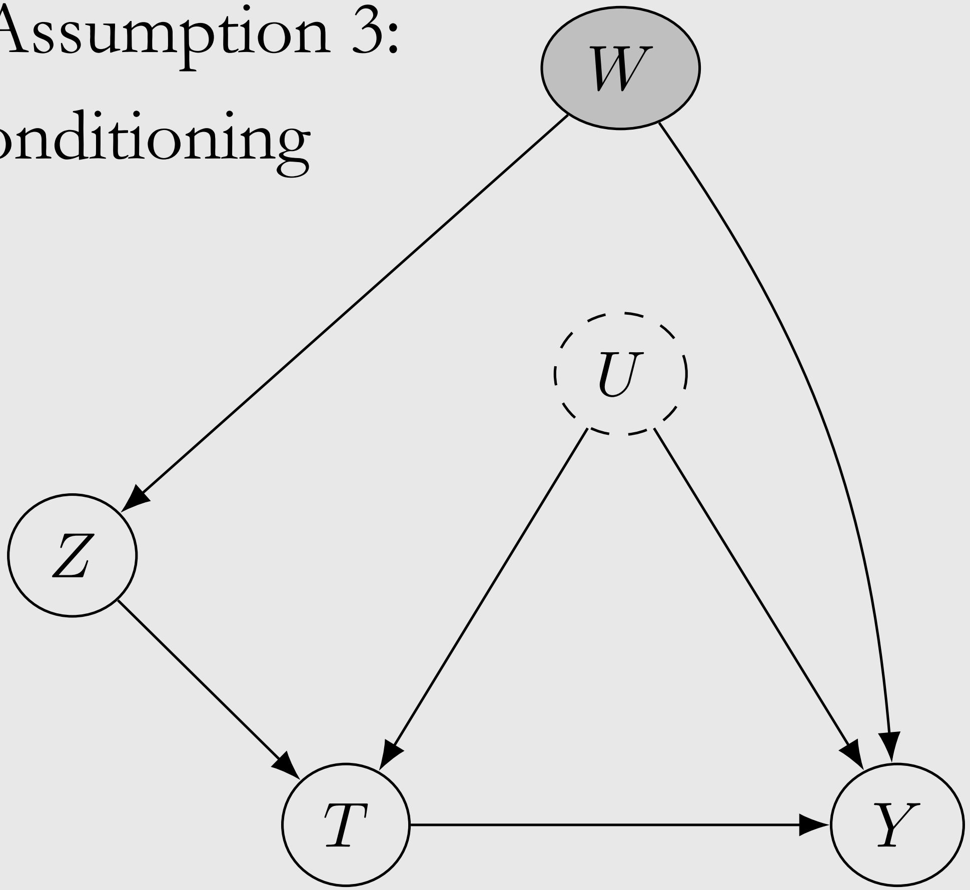


Conditional Instruments

Slightly weaker version of Assumption 3:

Unconfoundedness after conditioning on observed variables



Question:
What are the 3 assumptions we need to say that a given variable is an instrument, and what do they correspond to graphically?

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Why didn't we see instruments in Week 5 Identification?

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Week 5 was about nonparametric identification

No Nonparametric Identification of the ATE

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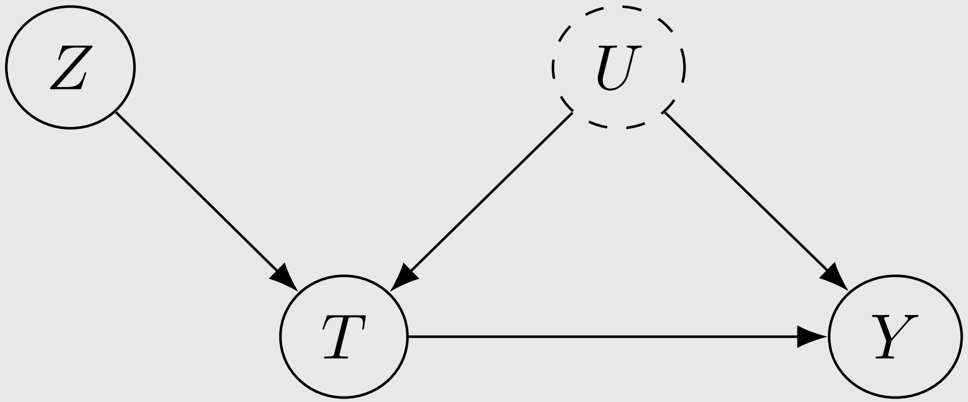
Week 5 was about nonparametric identification

Recall necessary condition for nonparametric identification:

For each backdoor path from T to any child that is an ancestor of Y, it is possible to block that path



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Another Assumption: Linear Outcome

$$Y := \delta T + \alpha_u U$$

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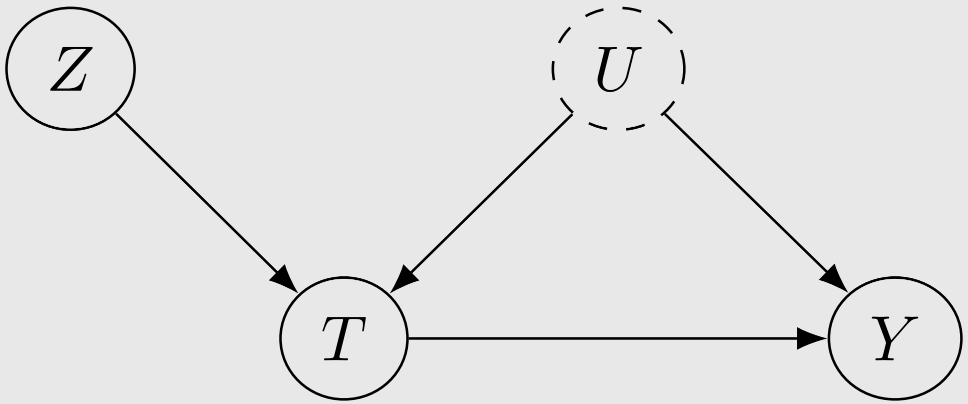
Z doesn't appear in this structural equation because of the exclusion restriction assumption



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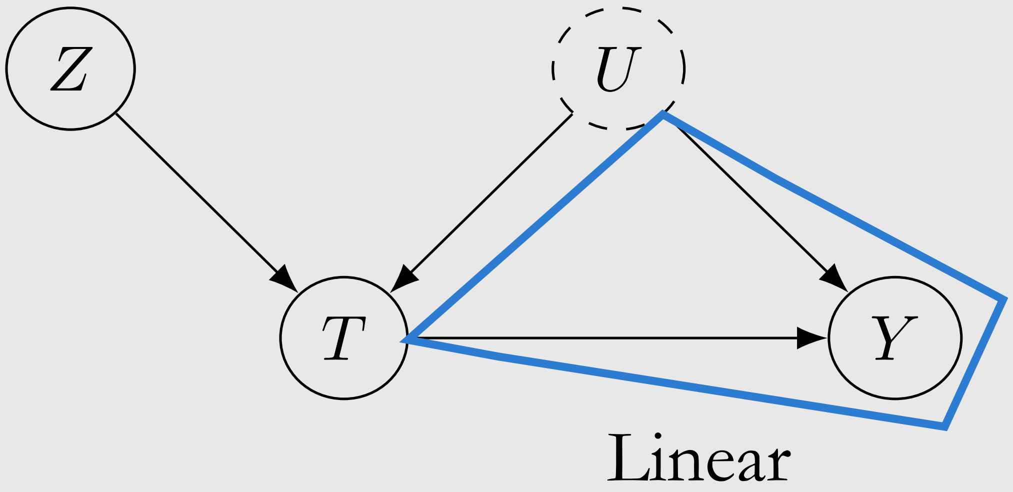
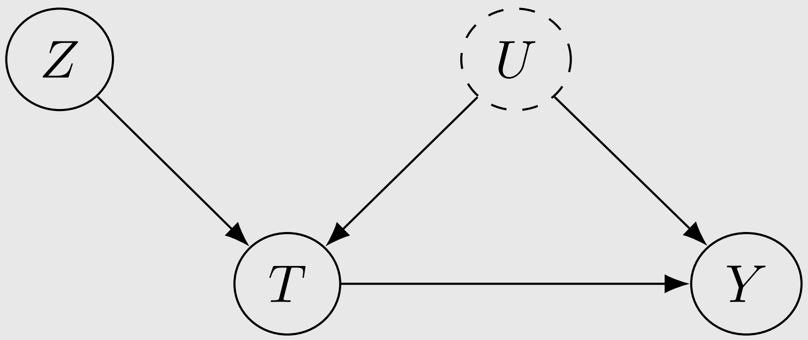


Warm-Up: Binary Linear Setting



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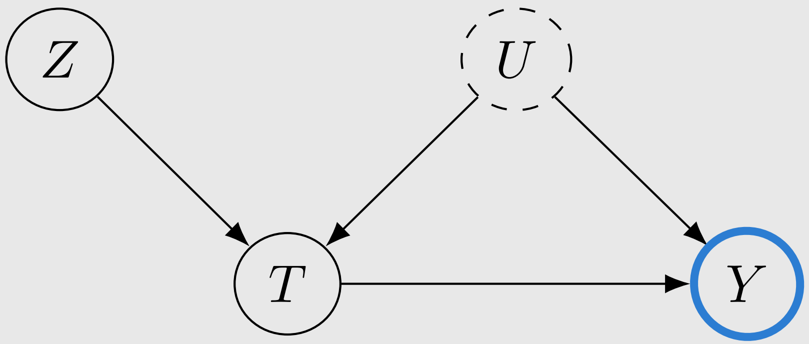

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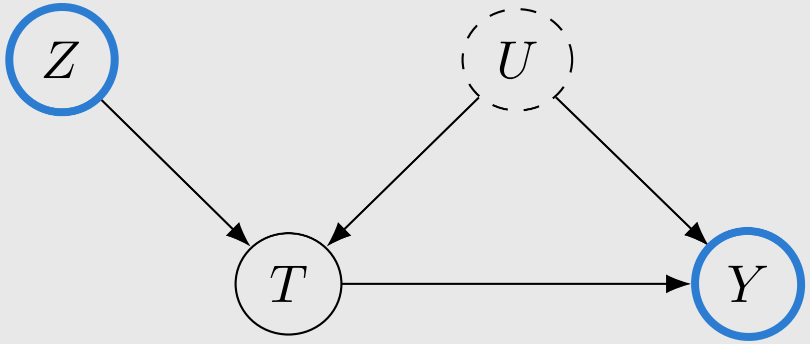

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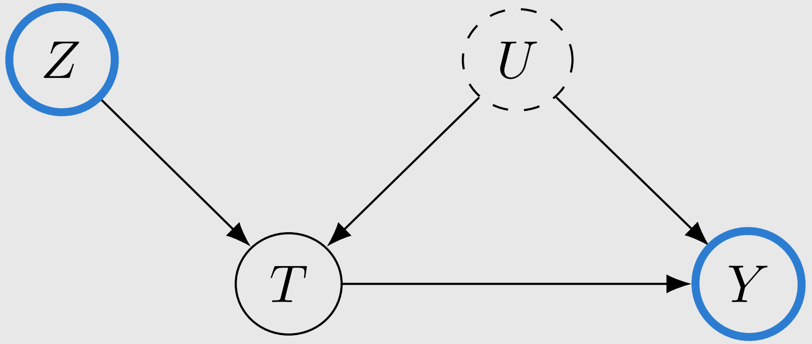
$\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0]$



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$$\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0]$$

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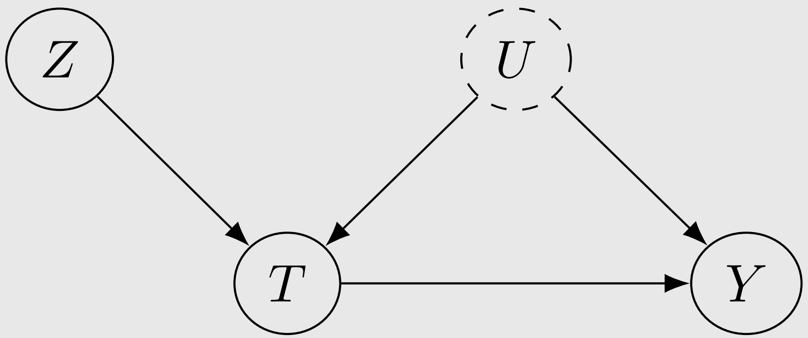

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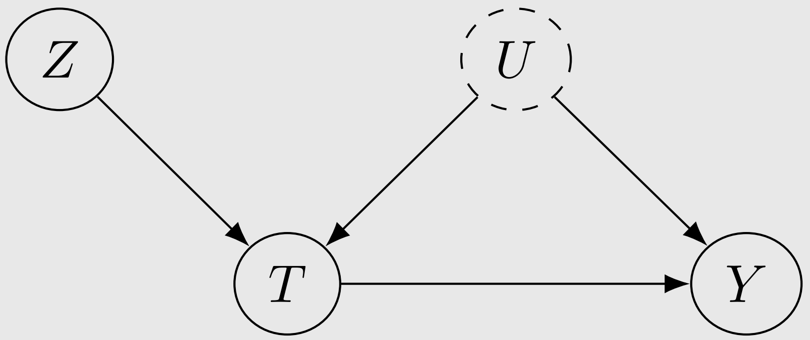
$$= \delta \left(\mathbb{E}[T \mid Z=1] - \mathbb{E}[T \mid Z=0]\right) + \alpha_u \left(\mathbb{E}[U \mid Z=1] - \mathbb{E}[U \mid Z=0]\right)$$



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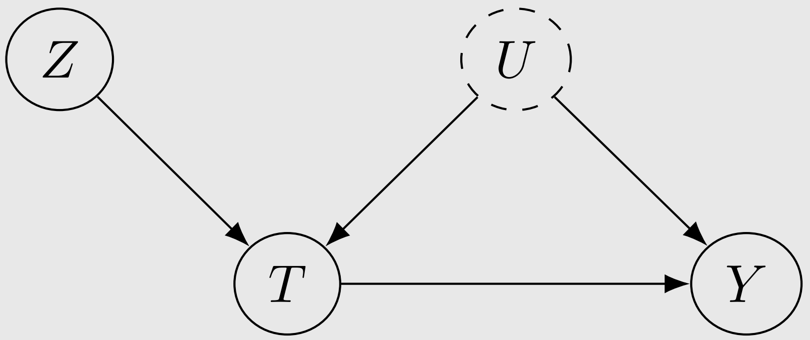
\end{aligned}$$



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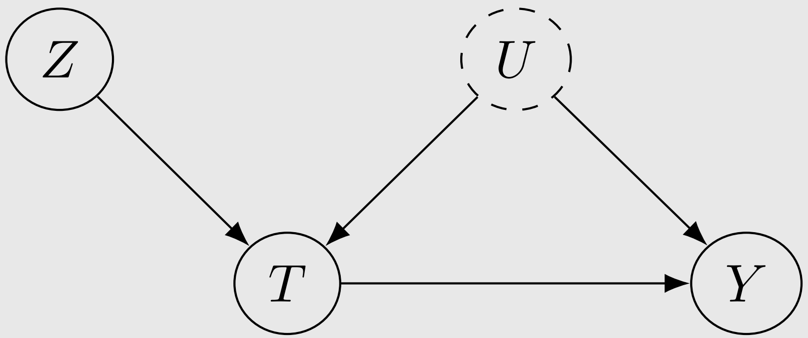

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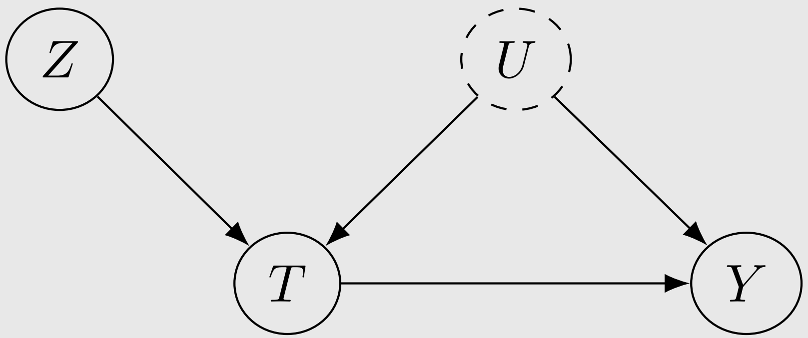

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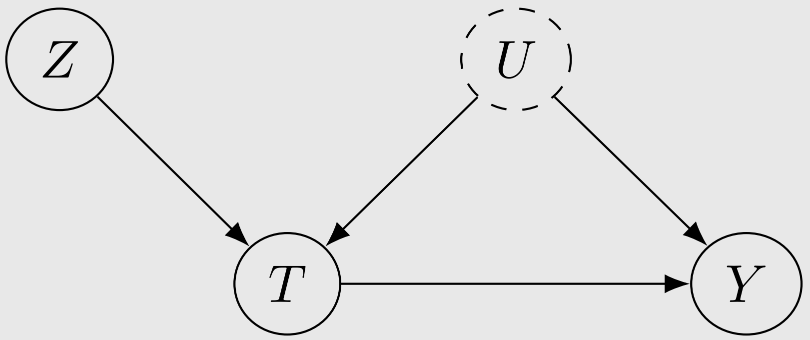
$$= \delta \left(\mathbb{E}[T \mid Z=1] - \mathbb{E}[T \mid Z=0]\right)$$

$$\delta = \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{\underbrace{\mathbb{E}[T \mid Z=1] - \mathbb{E}[T \mid Z=0]}_{\text{(non-zero due to relevance assumption)}}}$$



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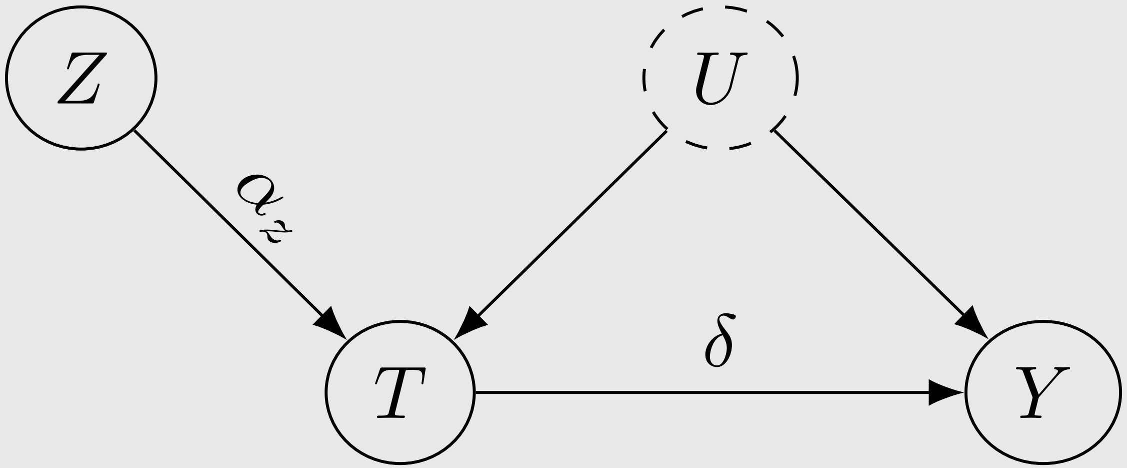
Multiplying Path Coefficients in Linear Setting



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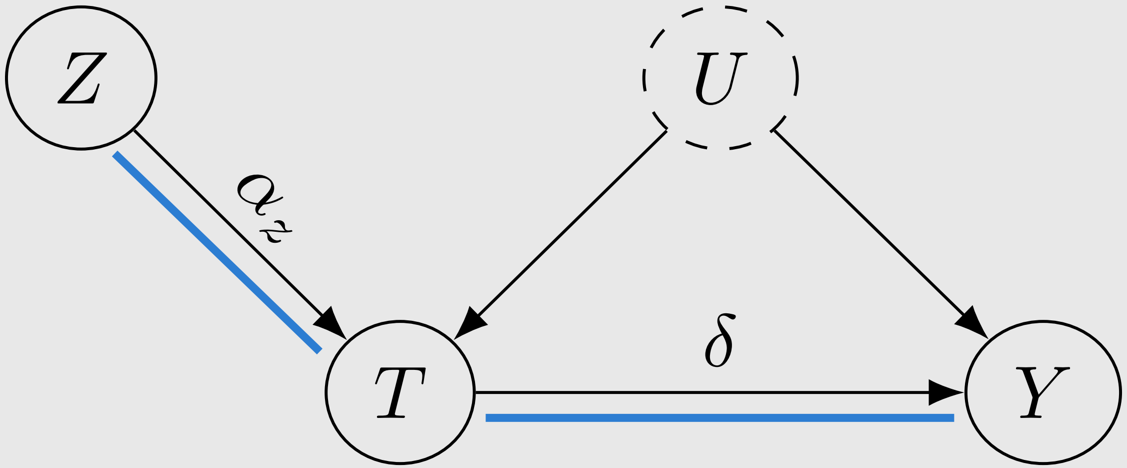

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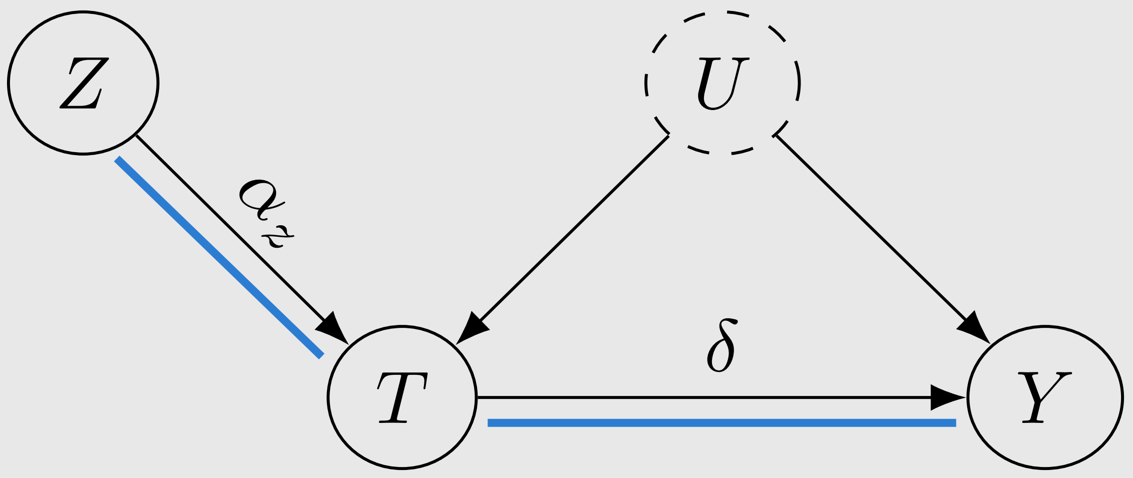

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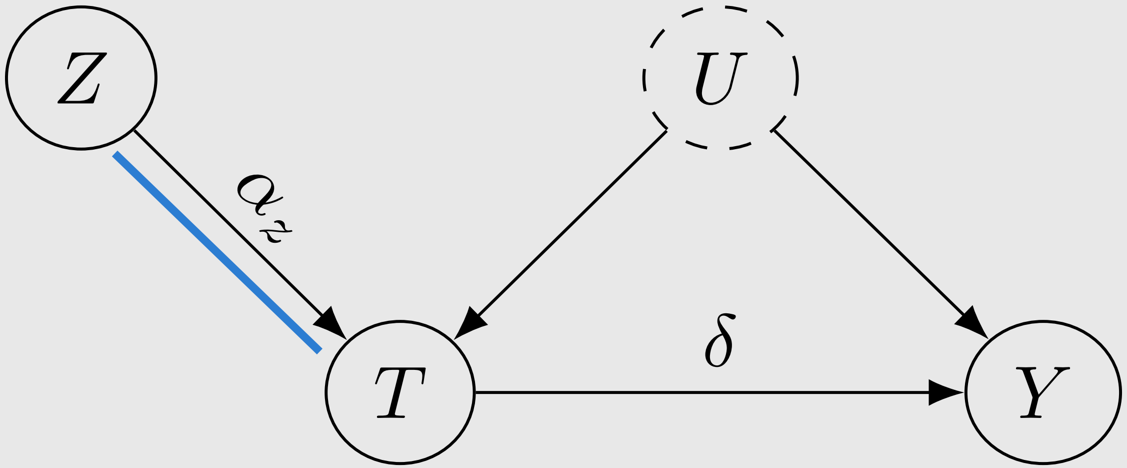

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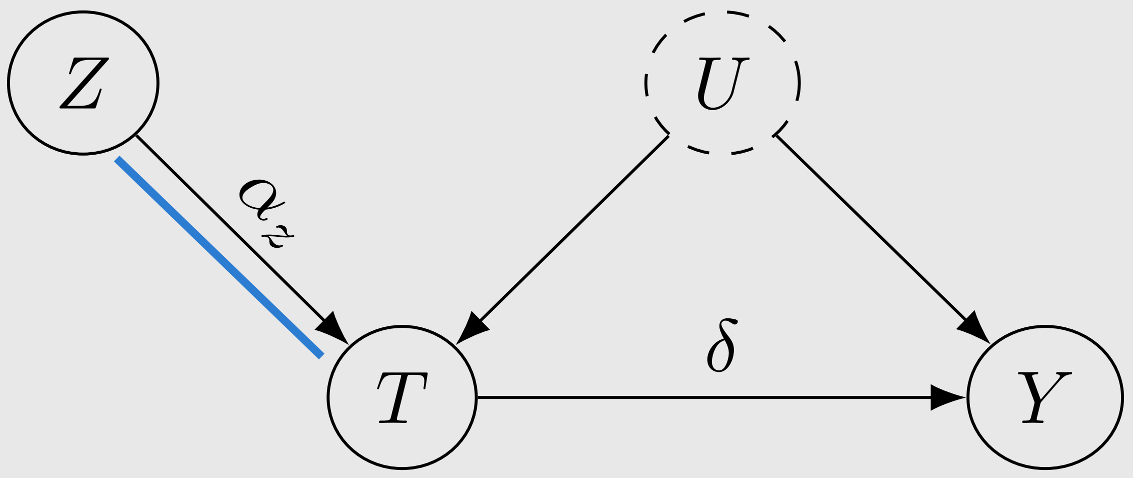

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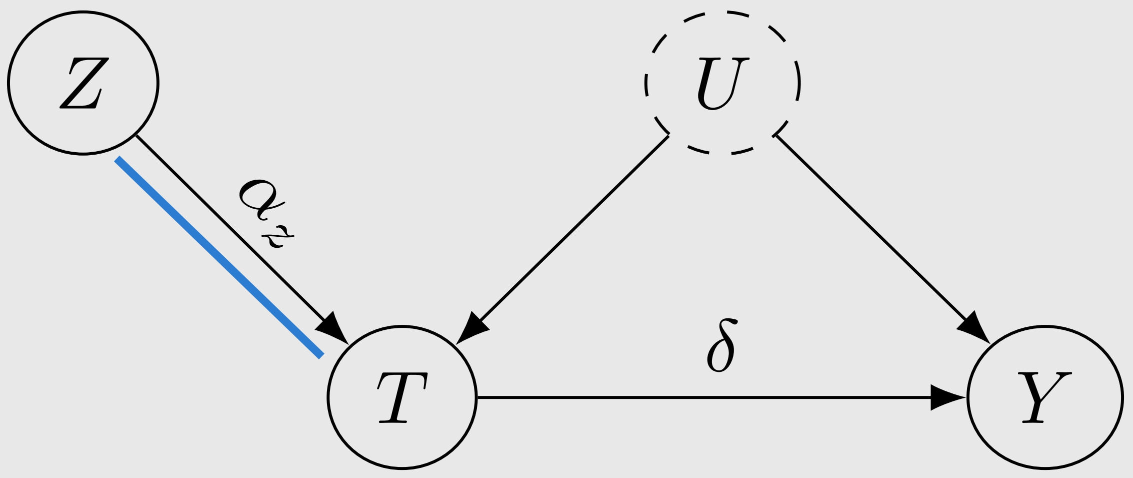

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Wald Estimator

Wald estimand:

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Wald estimator:

$$\hat{\delta} = \frac{\frac{1}{n_1} \sum_{i:z_i=1} Y_i - \frac{1}{n_0} \sum_{i:z_i=0} Y_i}{\frac{1}{n_1} \sum_{i:z_i=1} T_i - \frac{1}{n_0} \sum_{i:z_i=0} T_i}$$

Continuous Linear Setting

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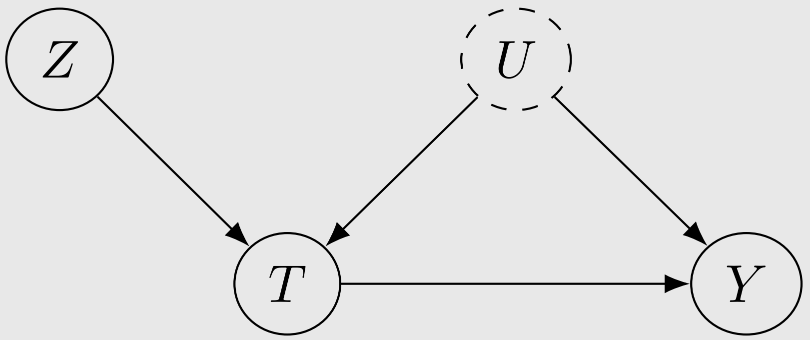

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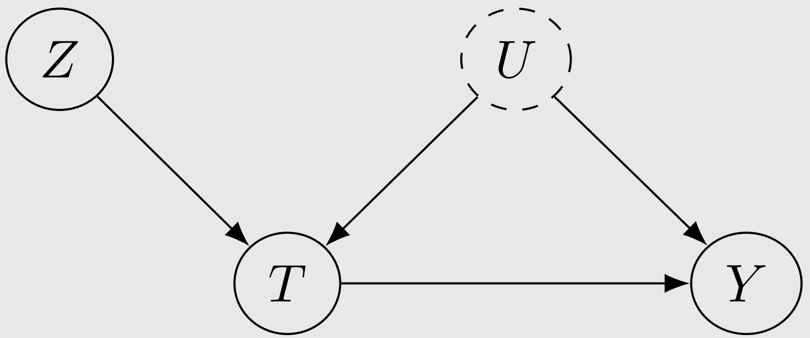
What if T and Z are continuous?



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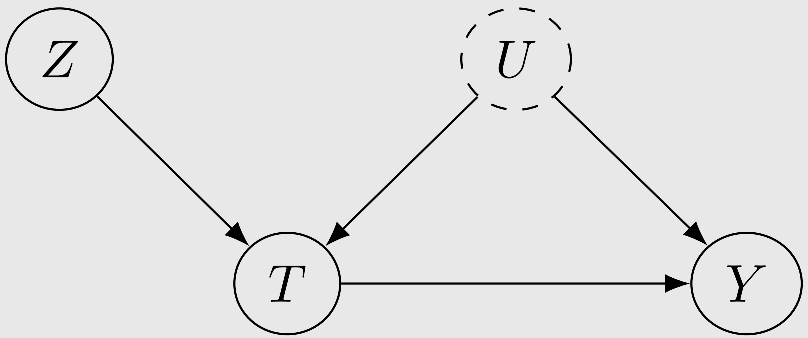
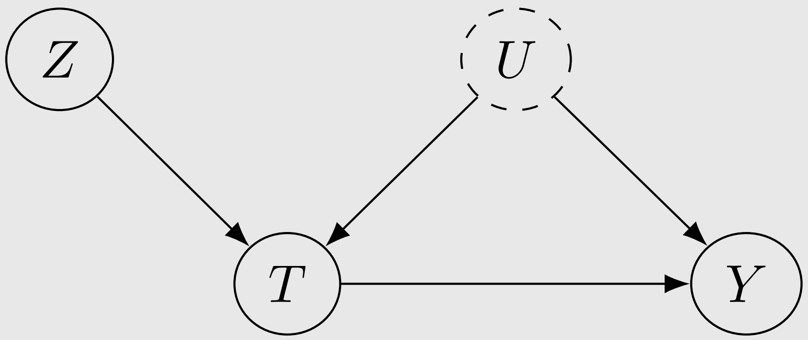
$$\delta = \frac{\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0]}{\mathbb{E}[T \mid Z = 1] - \mathbb{E}[T \mid Z = 0]} \qquad \delta = \frac{\mathrm{Cov}(Y, Z)}{\mathrm{Cov}(T, Z)}$$

What if T and Z are continuous?



$$Y := \delta T + \alpha_u U$$

Continuous Linear Setting



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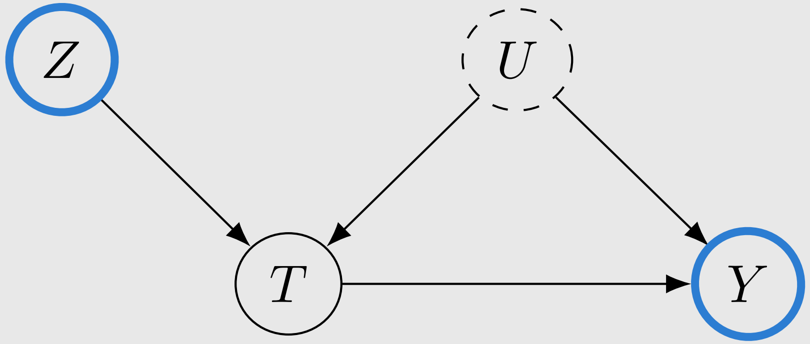

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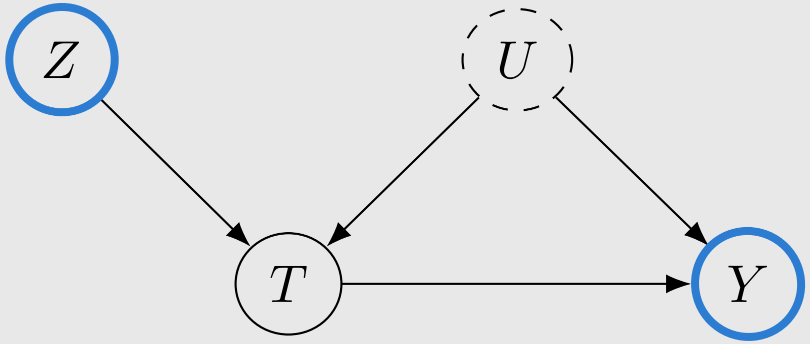
Continuous Linear Setting

$\text{Cov}(Y, Z)$



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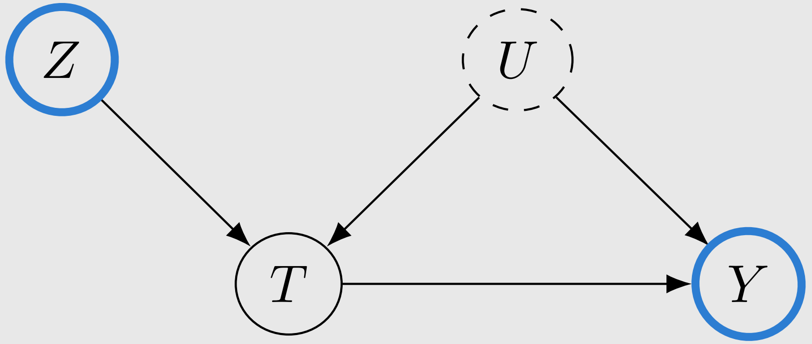
Continuous Linear Setting

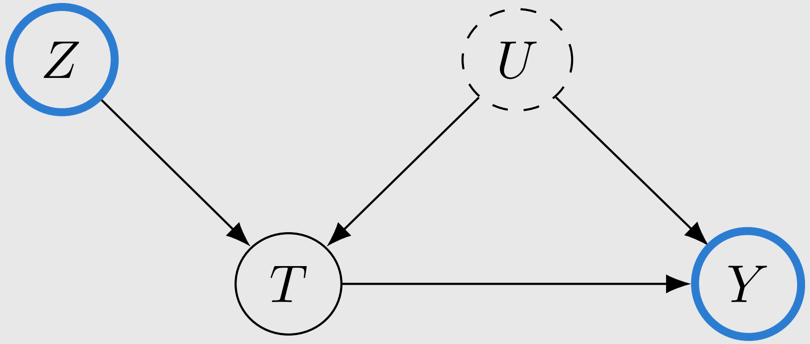
$\text{Cov}(Y, Z) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z]$



$$Y := \delta T + \alpha_u U$$

Continuous Linear Setting

$$
\begin{aligned}
\mathrm{Cov}(Y, Z) &= \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \\
&= \mathbb{E}[(\delta T + \alpha_u U)Z] - \mathbb{E}[\delta T + \alpha_u U]\mathbb{E}[Z] \qquad \text{(exclusion restriction and linear outcome assumptions)}
\end{aligned}
$$



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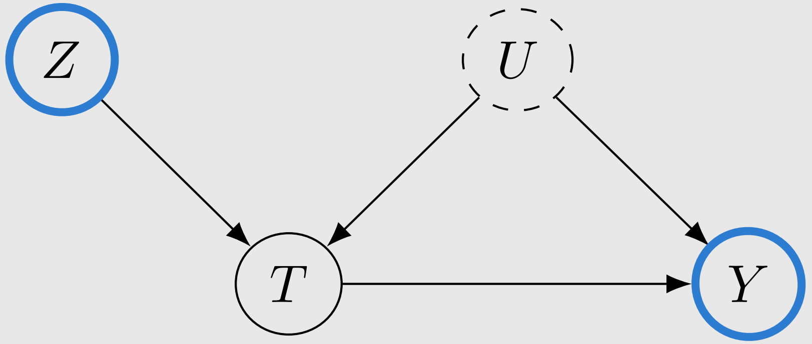
\end{aligned}
$$



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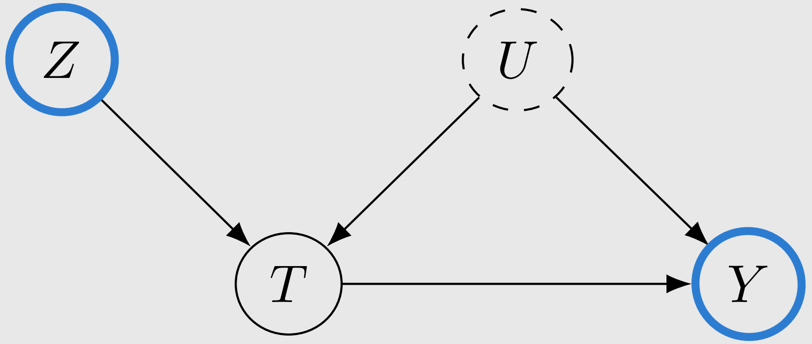
\end{aligned}
$$



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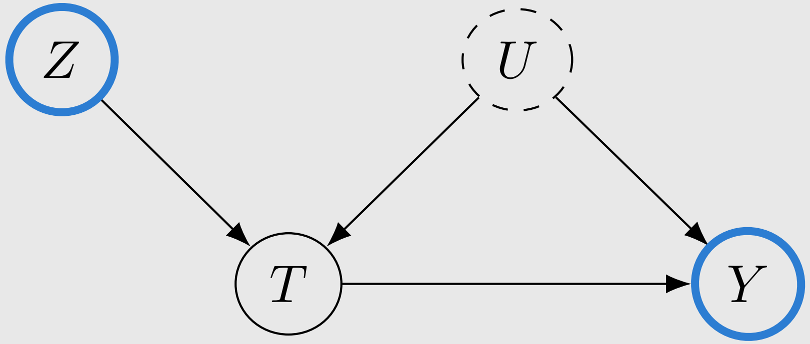
$$



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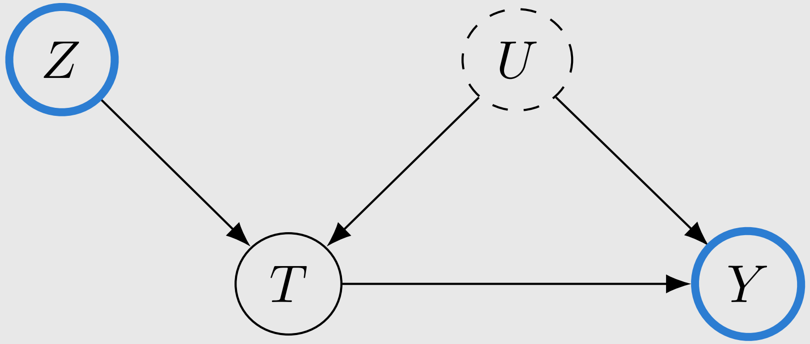

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\end{aligned}$$

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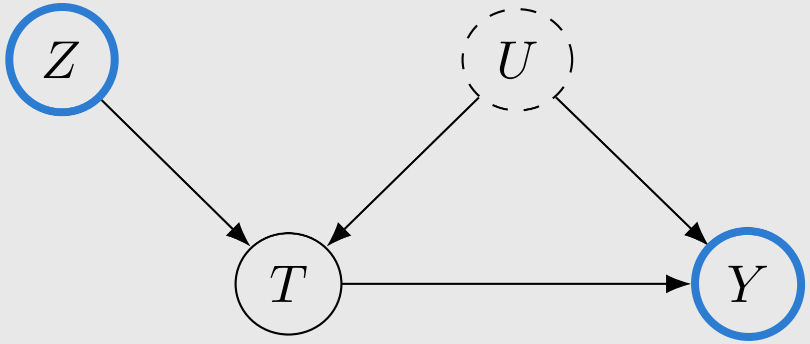

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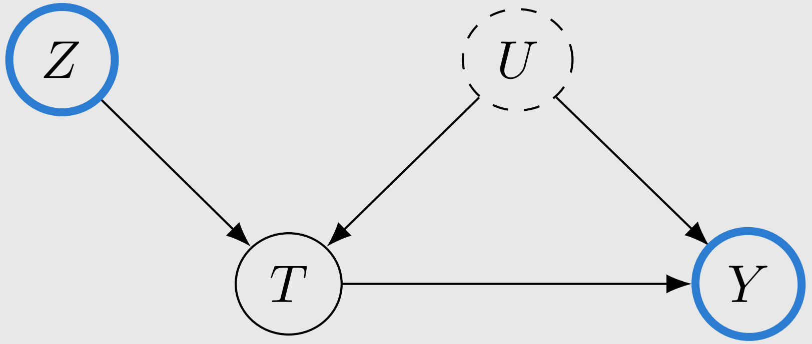
&= \delta \text{Cov}(T, Z) \qquad \text{(instrumental unconfoundedness assumption)}
\end{aligned}
$$

$$
\delta = \frac{\text{Cov}(Y, Z)}{\underbrace{\text{Cov}(T, Z)}_{\text{(non-zero due to relevance assumption)}}}
$$



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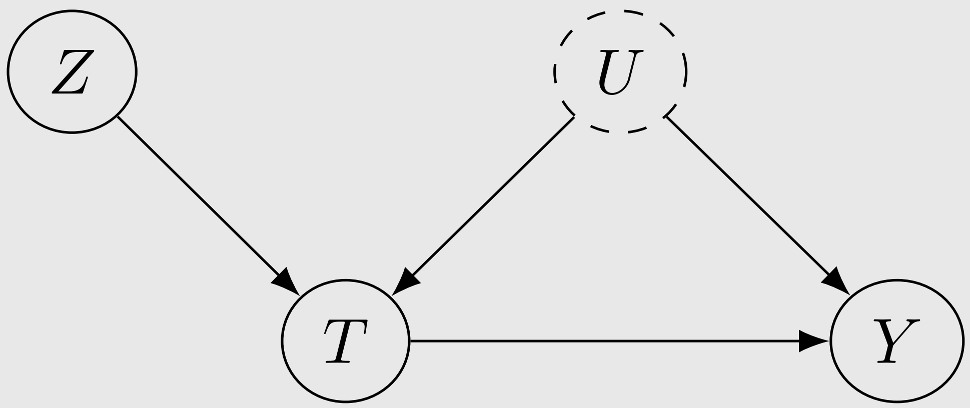
Continuous Linear Setting: Estimator 1

$$\delta = \frac{\mathrm{Cov}(Y, Z)}{\mathrm{Cov}(T, Z)}$$

Continuous Linear Setting: Estimator 1

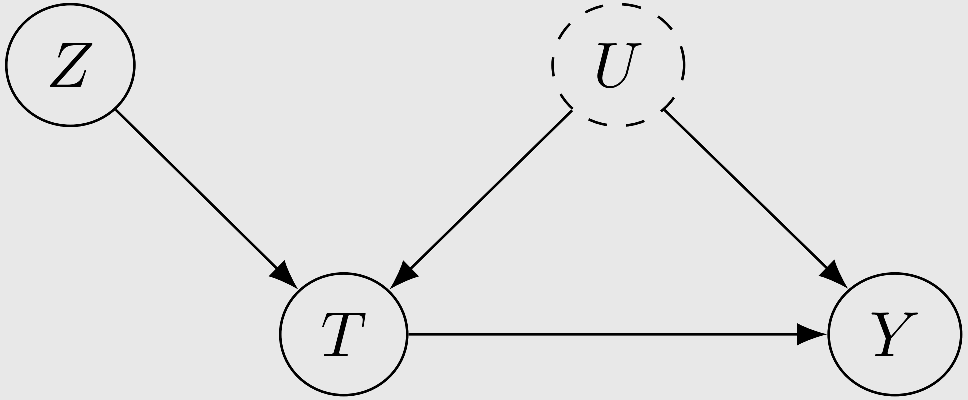
$$\hat{\delta} = \frac{\widehat{\text{Cov}}(Y, Z)}{\widehat{\text{Cov}}(T, Z)}$$

Two-Stage Least Squares Estimator



Two-Stage Least Squares Estimator

1. Linearly regress T on Z to estimate $\mathbb{E}[T \mid Z]$. This gives us the projection of T onto Z: $\hat{T}$



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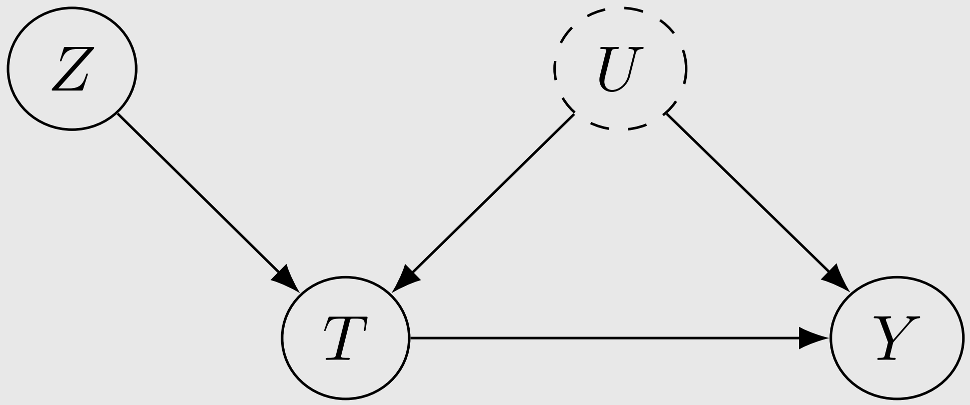
2. Linearly regress Y on $\hat{T}$ to estimate $\mathbb{E}[Y \mid \hat{T}]$. Obtain our estimate $\hat{\delta}$ as the fitted coefficient in front of $\hat{T}$.



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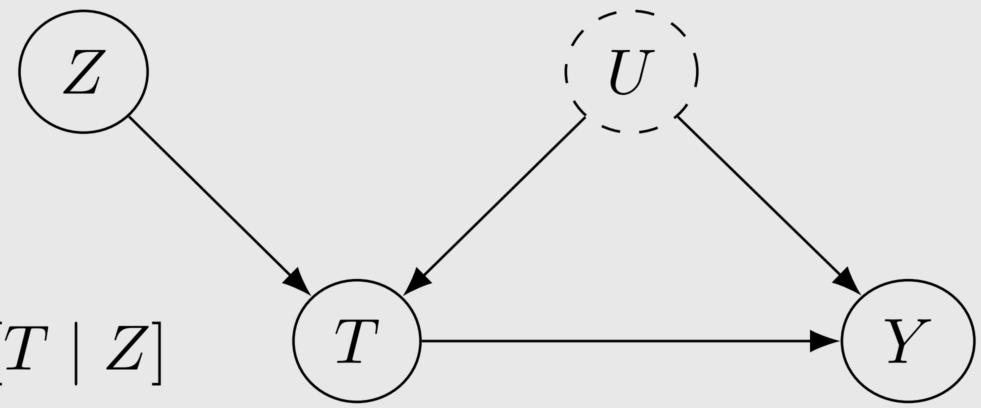

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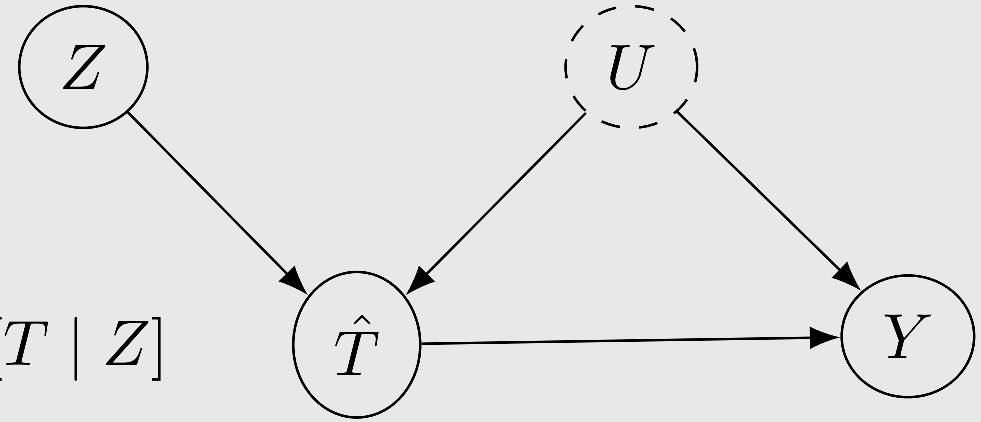

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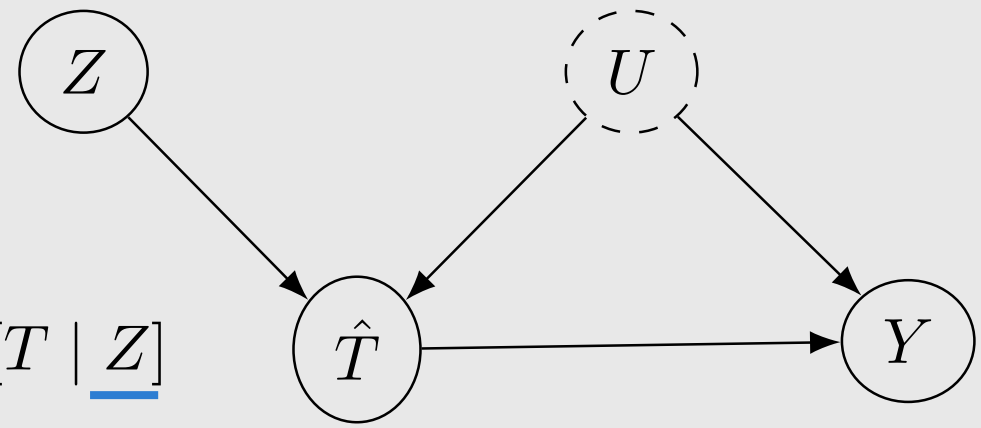

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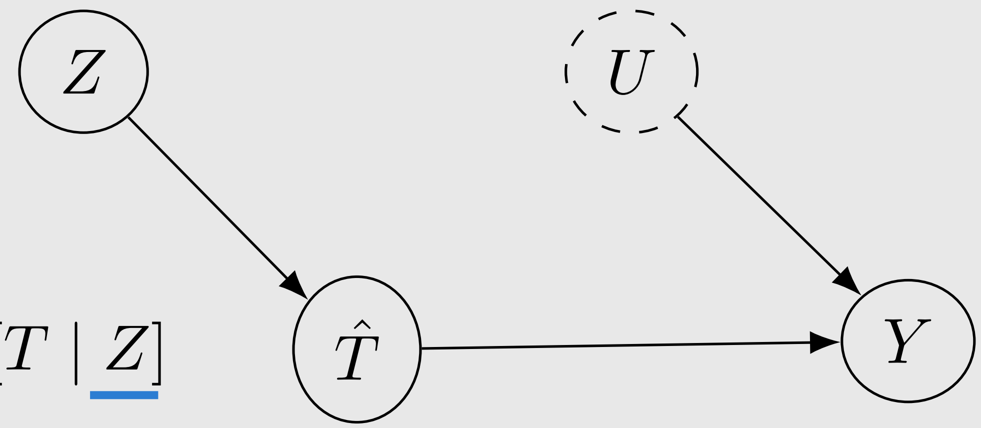

Two-Stage Least Squares Estimator

1. Linearly regress T on Z to estimate $\mathbb{E}[T \mid Z]$. This gives us the projection of T onto Z: $\hat{T}$
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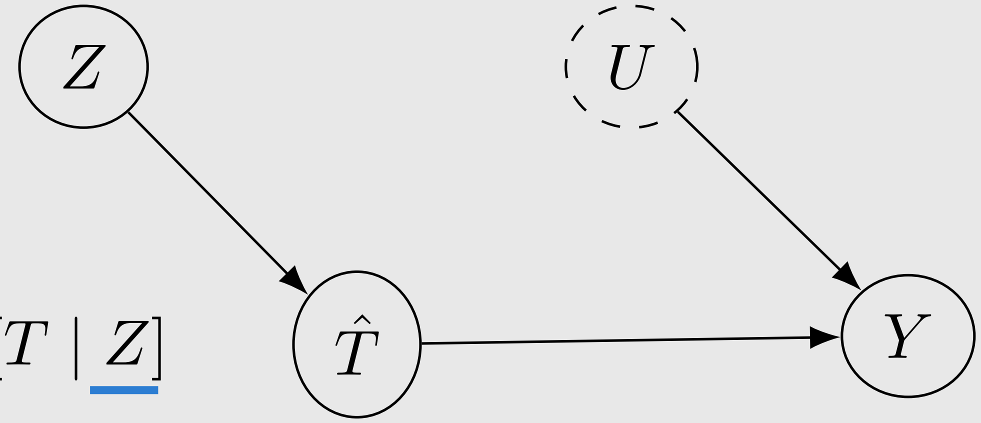
Also works as an estimator in the binary setting

$\hat{T} = \hat{\mathbb{E}}[T \mid Z]$



Question:
In the binary linear, setting where is each assumption used in the proof below?

$$
\begin{aligned}
&\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0] \\
&\quad = \mathbb{E}[\delta T + \alpha_u U \mid Z = 1] - \mathbb{E}[\delta T + \alpha_u U \mid Z = 0] \\
&\quad = \delta \left(\mathbb{E}[T \mid Z = 1] - \mathbb{E}[T \mid Z = 0]\right) + \alpha_u \left(\mathbb{E}[U \mid Z = 1] - \mathbb{E}[U \mid Z = 0]\right) \\
&\quad = \delta \left(\mathbb{E}[T \mid Z = 1] - \mathbb{E}[T \mid Z = 0]\right) + \alpha_u \left(\mathbb{E}[U] - \mathbb{E}[U]\right) \\
&\quad = \delta \left(\mathbb{E}[T \mid Z = 1] - \mathbb{E}[T \mid Z = 0]\right)
\end{aligned}
$$

$$
\delta = \frac{\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0]}{\mathbb{E}[T \mid Z = 1] - \mathbb{E}[T \mid Z = 0]}
$$

Question:
In the continuous linear setting, prove the following:

$$\delta = \frac{\mathrm{Cov}(Y, Z)}{\mathrm{Cov}(T, Z)}$$

What is an Instrument?

No Nonparametric Identification of the ATE

Warm-Up: Linear Setting

Nonparametric Identification of Local ATE

More General Settings for the ATE

Linear Outcome Assumption as Homogeneity

Linear outcome assumption: $Y := \delta T + \alpha_u U$

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There are other variants of the linear outcome assumption that all require the treatment effect to be homogeneous (the same for all units) in some way (see, e.g., Section 16.3 of Hernán & Robins (2020))

Linear Outcome Assumption as Homogeneity

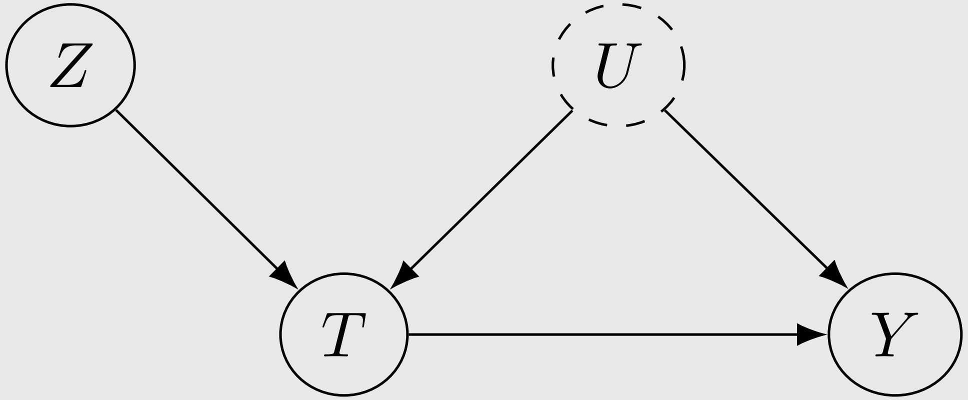
Linear outcome assumption: $Y := \delta T + \alpha_u U$

There are other variants of the linear outcome assumption that all require the treatment effect to be homogeneous (the same for all units) in some way (see, e.g., Section 16.3 of Hernán & Robins (2020))

Very restricting!

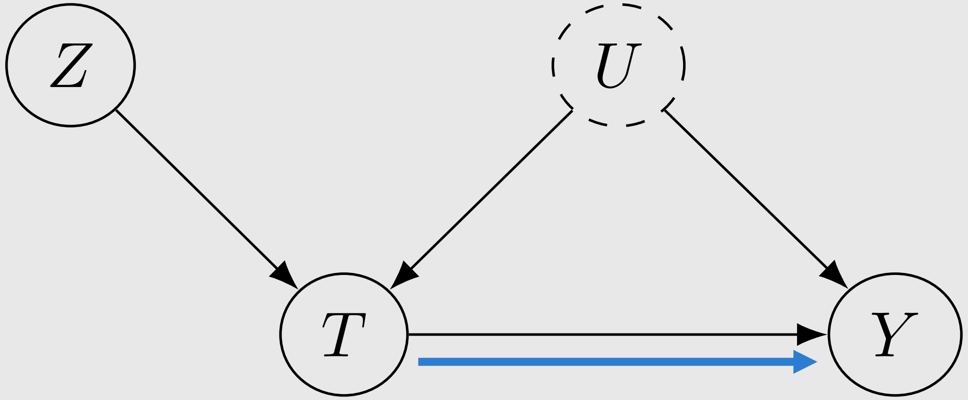
Can we get identification without parametric assumptions?

Potential Outcomes Notation with Instruments



Potential Outcomes Notation with Instruments

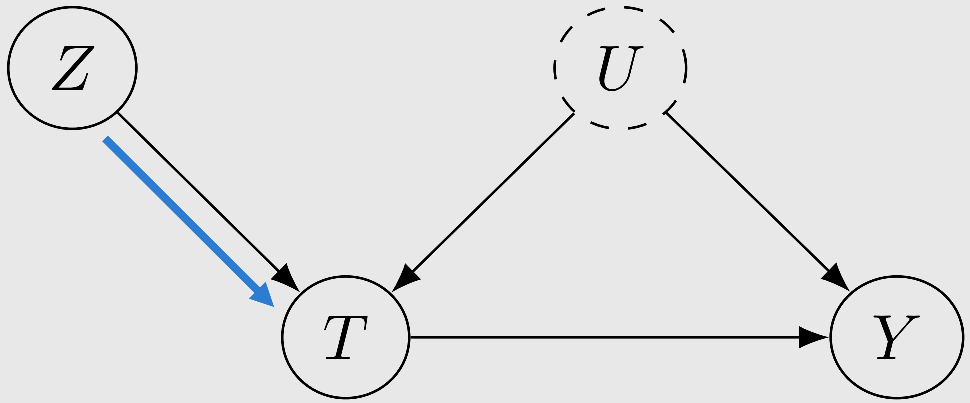
$Y(1)$ and $Y(0)$ are short for $Y(T = 1)$ and $Y(T = 0)$



Potential Outcomes Notation with Instruments

$Y(1)$ and $Y(0)$ are short for $Y(T = 1)$ and $Y(T = 0)$

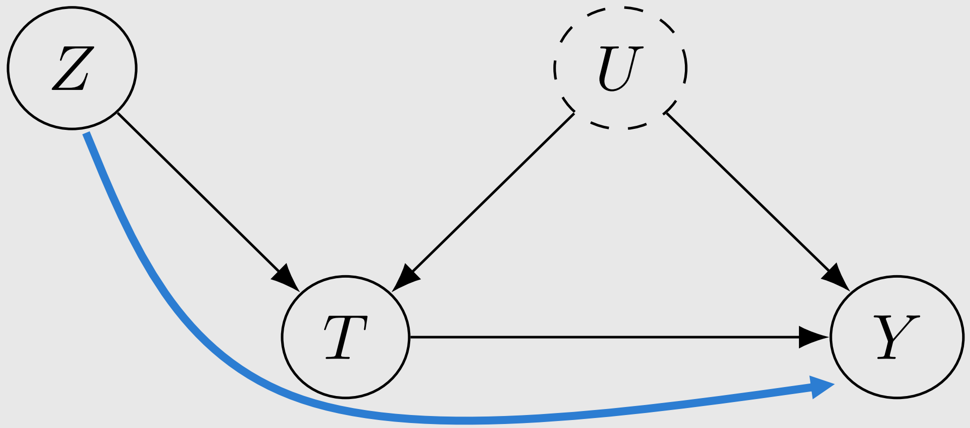
We now have $T(Z = 1)$ and $T(Z = 0)$ or $T(1)$ and $T(0)$ for short



Potential Outcomes Notation with Instruments

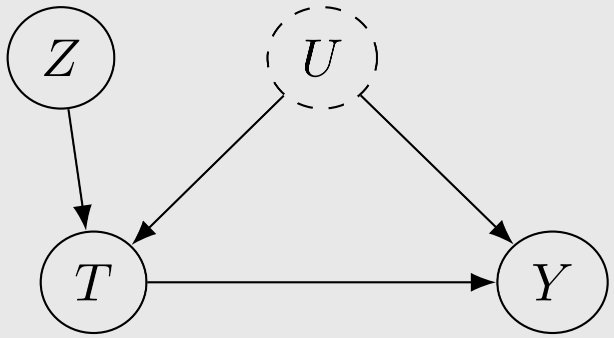
$Y(1)$ and $Y(0)$ are short for $Y(T=1)$ and $Y(T=0)$
We also have $Y(Z=1)$ and $Y(Z=0)$

We now have $T(Z=1)$ and $T(Z=0)$ or $T(1)$ and $T(0)$ for short



Principal Strata

Break data into 4 strata (groups) based on how the instrument affects the treatment they take



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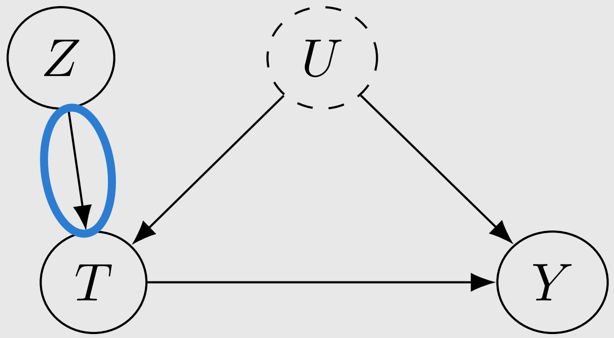
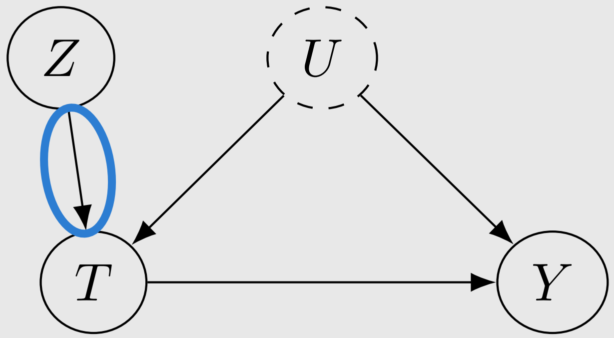


Principal Strata

Break data into 4 strata (groups) based on how the instrument affects the treatment they take

- Compliers: $T(Z=1)=1\,,\quad T(Z=0)=0$



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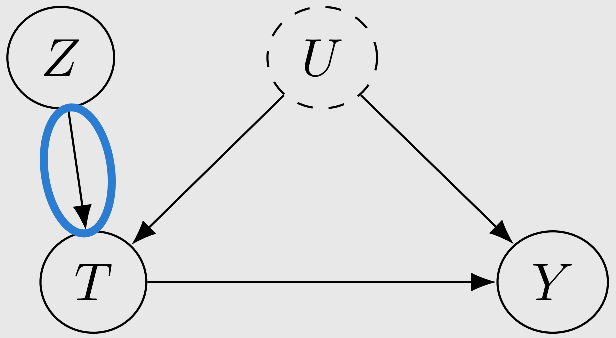

- Compliers: $T(Z=1)=1\,,\quad T(Z=0)=0$
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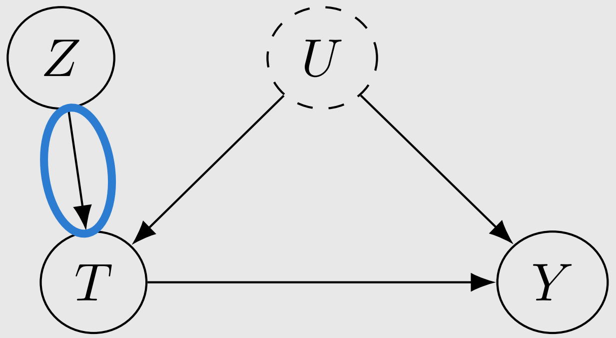

- Compliers: $T(Z=1)=1\,,\quad T(Z=0)=0$
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- Always-takers: $T(Z=1)=1\,,\quad T(Z=0)=1$

Principal Strata

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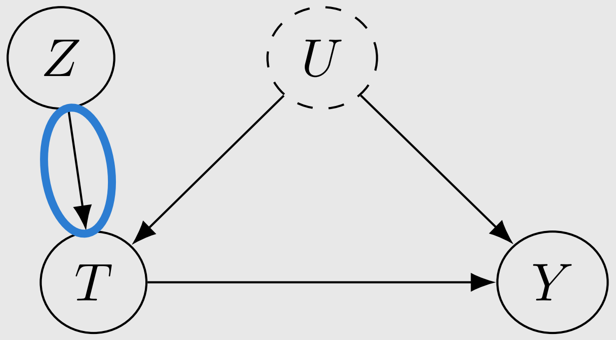

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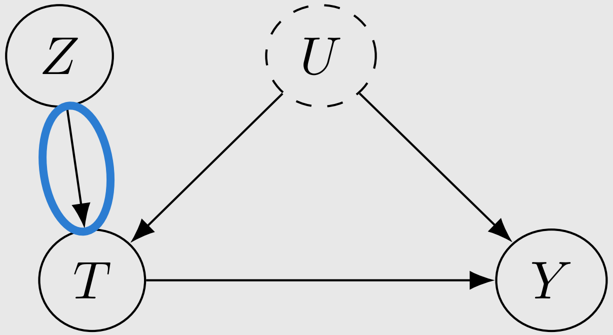

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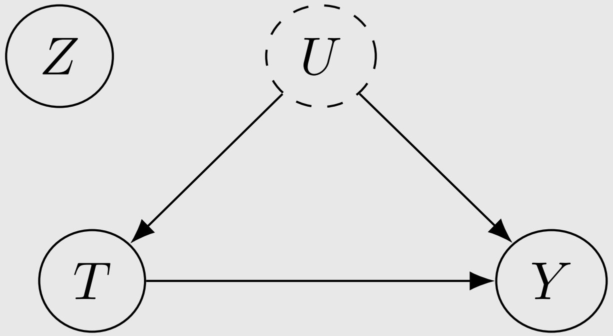

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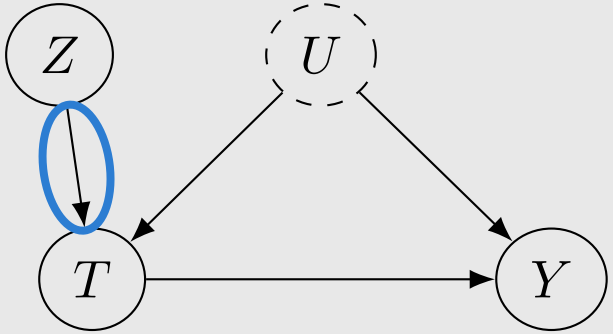

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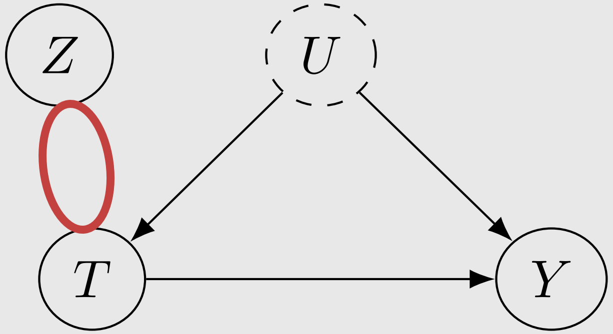
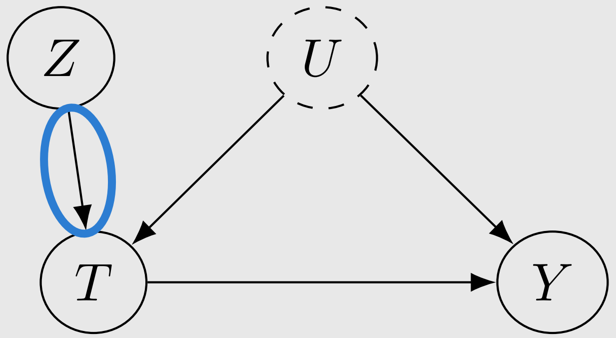


Monotonicity Assumption (No Defiers)

$$\forall i, \quad T_i(Z=1) \geq T_i(Z=0)$$

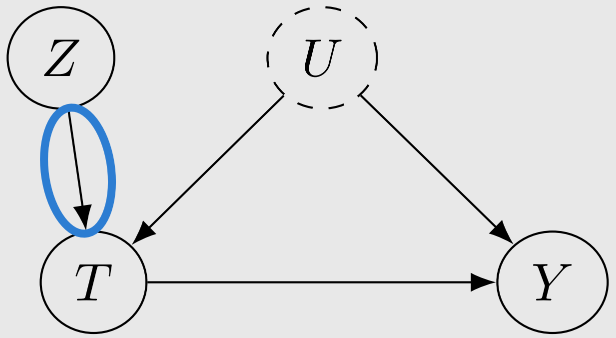
Monotonicity Assumption (No Defiers)

$$\forall i, \quad T_i(Z=1) \geq T_i(Z=0)$$



Monotonicity Assumption (No Defiers)

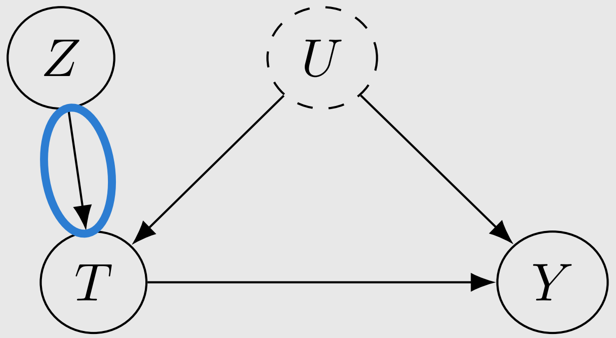
$$\forall i, \quad T_i(Z = 1) \geq T_i(Z = 0)$$



- Compliers: $T(Z = 1) = 1\,, \quad T(Z = 0) = 0$
- Defiers: $T(Z = 1) = 0\,, \quad T(Z = 0) = 1$
- Always-takers: $T(Z = 1) = 1\,, \quad T(Z = 0) = 1$
- Never-takers: $T(Z = 1) = 0\,, \quad T(Z = 0) = 0$

Monotonicity Assumption (No Defiers)

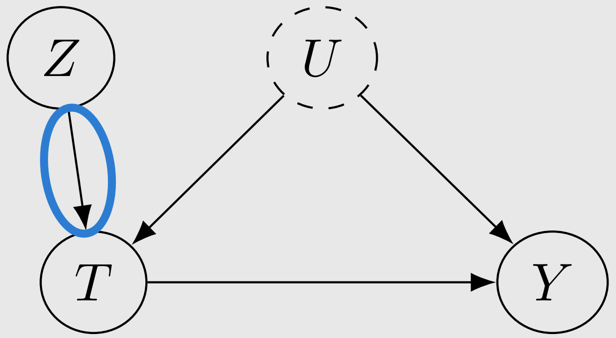
$$\forall i, \quad T_i(Z=1) \geq T_i(Z=0)$$



- Compliers: $T(Z=1)=1\,, \quad T(Z=0)=0$
- ~~Defiers: $T(Z=1)=0\,, \quad T(Z=0)=1$~~
- Always-takers: $T(Z=1)=1\,, \quad T(Z=0)=1$
- Never-takers: $T(Z=1)=0\,, \quad T(Z=0)=0$

Monotonicity Assumption (No Defiers)

$$\forall i, \quad T_i(Z=1) \geq T_i(Z=0)$$



- Compliers: $T(Z=1)=1\,, \quad T(Z=0)=0$
- ~~Defiers: $T(Z=1)=0\,, \quad T(Z=0)=1$~~
- Always-takers: $T(Z=1)=1\,, \quad T(Z=0)=1$
- Never-takers: $T(Z=1)=0\,, \quad T(Z=0)=0$

Deriving Local ATE Identification

with Monotonicity Assumption

Deriving Local ATE Identification

with Monotonicity Assumption

$$\mathbb{E}[Y(Z=1) - Y(Z=0)]$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)}
\end{aligned}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, \underline{P(T(1)=0, T(0)=1)} && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)}
\end{aligned}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, \underline{P(T(1)=0, T(0)=1)} && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)}
\end{aligned}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$\mathbb{E}[Y(Z=1) - Y(Z=0)]$$

$$= \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) \qquad \text{(Compliers)}$$

$$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) \qquad \text{(Defiers)}$$

$$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) \qquad \text{(Always-takers)}$$

$$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) \qquad \text{(Never-takers)}$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad \sout{+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1)} && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)}
\end{aligned}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$\mathbb{E}[Y(Z=1) - Y(Z=0)]$$

$= \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)$ (Compliers)

~~$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1)$ (Defiers)~~

~~$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1)$ (Always-takers)~~

~~$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0)$ (Never-takers)~~

Deriving Local ATE Identification

with Monotonicity Assumption

$\mathbb{E}[Y(Z=1) - Y(Z=0)]$

$= \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)$ (Compliers)

~~$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1)$ (Defiers)~~

~~$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1)$ (Always-takers)~~

~~$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0)$ (Never-takers)~~

$= \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)$

Deriving Local ATE Identification

with Monotonicity Assumption

$$\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}$$

$$\mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0] = \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\quad\quad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\quad\quad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\quad\quad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

$$
\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

Local ATE (LATE) or Complier average causal effect (CACE):

$$\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

Local ATE (LATE) or Complier average causal effect (CACE):

$$
\begin{aligned}
\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] &= \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)} \\
&= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)}
\end{aligned}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

Local ATE (LATE) or Complier average causal effect (CACE): $\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \dfrac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}$

$$= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)}$$

$P(T(1)=1, T(0)=0)$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

Local ATE (LATE) or Complier average causal effect (CACE):

$$
\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)} = \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)}
$$

$P(T(1)=1, T(0)=0)$

Deriving Local ATE Identification

with Monotonicity Assumption

$$\mathbb{E}[Y(Z=1) - Y(Z=0)]$$

$$= \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) \quad \text{(Compliers)}$$

$$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) \quad \text{(Defiers)}$$

$$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) \quad \text{(Always-takers)}$$

$$+ \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) \quad \text{(Never-takers)}$$

$$= \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)$$

Local ATE (LATE) or Complier average causal effect (CACE): $\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \dfrac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}$

$$P(T(1)=1, T(0)=0) = 1 - P(T=0 \mid Z=1) - P(T=1 \mid Z=0)$$

$$= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)}$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}$$

Local ATE (LATE) or Complier average causal effect (CACE): $\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \dfrac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}$

$$= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)}$$

$$\begin{aligned}
P(T(1)=1, T(0)=0) &= 1 - P(T=0 \mid Z=1) - P(T=1 \mid Z=0) \\
&= 1 - (1 - P(T=1 \mid Z=1)) - P(T=1 \mid Z=0)
\end{aligned}$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

Local ATE (LATE) or Complier average causal effect (CACE):

$$
\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}
$$

$$
= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)}
$$

$$
\begin{aligned}
P(T(1)=1, T(0)=0) &= 1 - P(T=0 \mid Z=1) - P(T=1 \mid Z=0) \\
&= 1 - (1 - P(T=1 \mid Z=1)) - P(T=1 \mid Z=0) \\
&= P(T=1 \mid Z=1) - P(T=1 \mid Z=0)
\end{aligned}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

Local ATE (LATE) or Complier average causal effect (CACE):

$$
\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)}
$$

$$
= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)}
$$

$$
\begin{aligned}
P(T(1)=1, T(0)=0) &= 1 - P(T=0 \mid Z=1) - P(T=1 \mid Z=0) \\
&= 1 - (1 - P(T=1 \mid Z=1)) - P(T=1 \mid Z=0) \\
&= P(T=1 \mid Z=1) - P(T=1 \mid Z=0) \\
&= \mathbb{E}[T \mid Z=1] - \mathbb{E}[T \mid Z=0]
\end{aligned}
$$

Deriving Local ATE Identification

with Monotonicity Assumption

$$
\begin{aligned}
&\mathbb{E}[Y(Z=1) - Y(Z=0)] \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0) && \text{(Compliers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=1]\, P(T(1)=0, T(0)=1) && \text{(Defiers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=1]\, P(T(1)=1, T(0)=1) && \text{(Always-takers)} \\
&\qquad + \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=0, T(0)=0]\, P(T(1)=0, T(0)=0) && \text{(Never-takers)} \\
&\quad = \mathbb{E}[Y(Z=1) - Y(Z=0) \mid T(1)=1, T(0)=0]\, P(T(1)=1, T(0)=0)
\end{aligned}
$$

Local ATE (LATE) or Complier average causal effect (CACE):

$$
\begin{aligned}
\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] &= \frac{\mathbb{E}[Y(Z=1) - Y(Z=0)]}{P(T(1)=1, T(0)=0)} \\
&= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{P(T(1)=1, T(0)=0)} \\
&= \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{\mathbb{E}[T \mid Z=1] - \mathbb{E}[T \mid Z=0]}
\end{aligned}
$$

$$
\begin{aligned}
P(T(1)=1, T(0)=0) &= 1 - P(T=0 \mid Z=1) - P(T=1 \mid Z=0) \\
&= 1 - (1 - P(T=1 \mid Z=1)) - P(T=1 \mid Z=0) \\
&= P(T=1 \mid Z=1) - P(T=1 \mid Z=0) \\
&= \mathbb{E}[T \mid Z=1] - \mathbb{E}[T \mid Z=0]
\end{aligned}
$$

Nonparametric Identification of LATE Under Monotonicity Assumption

Local ATE (LATE) or complier average causal effect (CACE):

$$\mathbb{E}[Y(T=1) - Y(T=0) \mid T(1)=1, T(0)=0] = \frac{\mathbb{E}[Y \mid Z=1] - \mathbb{E}[Y \mid Z=0]}{\mathbb{E}[T \mid Z=1] - \mathbb{E}[T \mid Z=0]}$$

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$\mathbb{E}[Y(T=1) - Y(T=0)]$ (ATE, for contrast)

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This is the Wald estimand!

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Problems:

Nonparametric Identification of LATE Under Monotonicity Assumption

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$\mathbb{E}[Y(T=1) - Y(T=0)]$ (ATE, for contrast)

Problems:

- Monotonicity isn't always satisfied

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$\mathbb{E}[Y(T=1) - Y(T=0)]$ (ATE, for contrast)

This is the Wald estimand!

Problems:

- Monotonicity isn't always satisfied
- Even if it is, we are usually interested in the average effect across the whole population (ATE), rather than just the compliers (CACE)

Question:
What causal estimand can we nonparametrically identify with an instrument and the monotonicity assumption?

What is an Instrument?

No Nonparametric Identification of the ATE

Warm-Up: Linear Setting

Nonparametric Identification of Local ATE

More General Settings for the ATE

Nonparametric Outcome with Additive Noise

$$Y := f(T, W) + U$$

Nonparametric Outcome with Additive Noise

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where f can be some very flexible model such as a deep neural network (see, e.g., Hartford et al. (2017), Xu et al. (2020), and references therein)

Nonparametric Outcome with Additive Noise

(Semi-parametric)

$$Y := f(T, W) \underline{+ U}$$

where f can be some very flexible model such as a deep neural network (see, e.g., Hartford et al. (2017), Xu et al. (2020), and references therein)

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(rather than point identification)

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See Section 8.2 of Pearl's Causality book

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See Kilbertus et al. (2020) and references therein